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STABILITY OF LYOPHOBIC COLLOIDS

PART II. STABILITY OF HYDROUS FERRIC OXIDE SOLS WITH SPECIAL REFERENCE TO THE DIELECTRIC CONSTANT OF THE MEDIUM

By

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ABSTRACT

The lowering of dielectric constant of the dispersing medium by adding some non-electrolyte to a lyophobic sol shows a decrease in its stability but this being not a general phenomenon, in this paper the changes in the dielectric constant of the dispersing medium water has been varied from 78·54 to 45·39 by the addition of dioxan to the hydrous ferric oxide sols, which were prepared by three different methods. The results show that the behaviour of these sols are not identical. While the sols of hydrous ferric oxide prepared by peptising the hydrous oxide either by dilute hydrochloric or acetic acid become unstable for all concentrations of dioxan, the sol prepared by Kreck's method is stablised by adding smaller concentrations of dioxan, but becomes unstable when the concentration of dioxan increases beyond 10%. These results indicate that the nature of hydrous ferric oxide considerably modifies the effect of non-electrolytes. It is well known that the coagulum of the hydrous oxide precipitated from Kreck's sol is more compact than those obtained from other two samples of the sols, where the coagulum is highly flocky and tends to form gel.

INTRODUCTION

The action of non-electrolytes on the stability of lyophobic colloids has largely been investigated.[1] Recently Ghosh and coworkers[2] have studied the effect of various non-electrolytes on the positively charged hydrous ferric oxide and negatively charged manganese dioxide sols. They have shown that besides the various factors, the free surface energy of the dispersed particles also guides the stability of a sol in the presence of some non-electrolytes. While many factors have been ascribed to affect the stability of a sol, *viz.*, the preferential adsorptive capacity of the peptising or coagulating ions, the amount of added electrolyte and its specific character, the chemical interaction between the coagulating and peptising ions, the dielectric constant[3] of the medium has also been emphasised to control the stability of a sol in the presence of a non-electrolyte. Freundlich[4] opined that the lowering of dielectric constant leads to the diminution of the charge on the particles and hence its stability. Keesom[5] also confirmed this view by noting stabilisation with the non-electrolytes which increase the dielectric constant of water. But Mukerji[6] from his cataphoretic measurements could not contribute to this view.

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In this investigation dioxan(diethylene dioxide) has been employed to lower the dielectric constant of the dispersing medium water and potassium chloride has been used as a coagulating electrolyte for the various sols of hydrous ferric oxide obtained by the different methods. It has been further observed that the effect of dioxan on the stability of sol is also controlled by the time of observation such that the effect becomes more pronounced when the time of observation is smaller. This had been neglected by previous workers.

EXPERIMENTAL

Three different samples of hydrous ferric oxide sol were prepared by peptising the hydrous oxide with hydrochloric and with acetic acids and by Kreck's method. These sols were nomenclatured as A, B and C respectively. The sol samples obtained by one method but having varying amount of peptising electrolyte are indicated by the suffix 1, 2 and 3. Iron content of the sols samples were kept the same (3·6298 gm of Fe^{+++}/litre) and their pH was adjusted to 2·2*, 3·6 and 4·5 by dialysing the sols to different extents. The purity of sols were determined by estimating the ratio of iron to chloride present in the system. Stability of these sols towards its complete coagulation by potassium chloride solution was determined. As the time of observation for coagulation greatly influences the precipitation value of an electrolyte, it was obtained graphically for infinite time. In order to avoid the variations of the values with temperature, the experiments were carried out in a precision theremostat. 2·0 ml of the sol was taken in each set of the test tubes. Different amounts of coagulation electrolyte made to 13·0 ml. with distilled water were taken in another set and kept in the thermostat till they attained the temperature of the bath. The contents of the two were mixed and left immersed in the thermostat undisturbed and the time of complete coagulation was noted at the point of visual separation of the clear liquid at the upper surface of the test tube. Similar studies were done in the presence of a known amount of the dioxan keeping the total volume the same, *i.e.* 15 ml. The dielectric constants of the mixtures of dioxan and distilled water were also measured but that of mixtures of electrolyte and sol could not be ascertained as measurement of the same in the presence of electrolytes is not ordinarily possible.

In the following tables we are reproducing some of our observations recorded at (30 ± 0·1)°C. In tables 1-9 the values of reciprocals of the time of coagulation are given for different amounts of coagulating electrolyte with varying % of dioxan.

TABLE 1

Sol sample A_1

(Purity of the sol $Fe^{+++}/Cl^- = 3{\cdot}6296/0{\cdot}96424 = 3{\cdot}7642$)

2·0N KCl (ml.)	$1/t \times 10^4$ in Sec.					
	Amount of Dioxan added					
	0%	5%	10%	20%	30%	40%
6·5	5·263	8·621	12·500	21·28	83·33	100·0
6·0	4·484	6·061	10·750	16·13	68·39	71·43
5·5	4·484	5·063	74·630	15·62	50·00	55·56
5·0	3·086	4·386	5·556	9·524	40·00	45·45
4·5	2·165	3·817	4·065	75·19	35·71	40·0
4·0	1·773	3·460	3·425	6·579	28·57	34·48
3·5	1·418	2·762	2·959	6·211	20·83	27·78
3·0	1·100	1·912	2.506	5·618	16·95	25·0
2·5	0·9156	1·548	1·869	4·405	13·89	21·28
2·0	0·8019	1·176	1·393	3·690	12·05	18·52

* pH of sol sample B_1 was 2·9 as it could not be obtained of lower pH value.

TABLE 2

Sol sample A_2

(Purity = 11.6348)

0·2N KCl (ml.)	$1/t \times 10^4$ in Sec. Amount of Dioxan added 0%	5%	10%	20%	30%	40%
6·5	3·546	8·772	14·29	43·48	86·96	200·0
6·0	2·801	7·042	11·11	31·25	71·43	166·7
5·5	2·50	5·495	9·709	27·03	52·63	142·9
5·0	2·053	4·566	75·19	19·05	41·67	111·1
4·5	1·862	3·472	66·23	15·15	35·71	90·91
4·0	1·555	2·817	62·50	12·50	28·57	71·43
3·5	1·250	2·506	46·51	10·00	22·94	55·56
3·0	0·9615	2·141	41·67	7·937	19·92	44·44
2·5	0·8621	1·949	28·57	68·03	16·95	37·04
2·0	0·6993	1·686	22·32	5·556	14·08	31·45

TABLE 3

Sol sample A_3

(Purity = 19 6897)

0·1N KCl (ml.)	$1/t \times 10^4$ in Sec. Amount of Dioxan added 0%	5%	10%	20%	30%	40%
6·5	2·618	3·086	3·802	6·250	22·22	62·50
6·0	1·972	2·506	3·155	5·236	20·00	52·63
5·5	1·701	1·972	2·667	4·630	17·86	42·55
5·0	1·399	1·618	2·123	3·906	14·71	34·48
4·5	1·096	1·404	1·770	3·378	11·79	27·78
4·0	0·8475	1·188	1·486	2·857	10·00	22·22
3·5	0·7246	1·000	1·233	2·591	8·197	18·18
3·0	0·6135	0·813	1·101	23·98	7·042	14·93
2·5	0·5291	0·6757	0·9119	1·9610	5·848	13·51
2 0	0·4587	0·5747	0·7780	1·623	5·00	11·76

TABLE 4

Sol sample B_1

2·0N KCl (ml.)	$1/t \times 10^4$ in Sec.					
	Amount of Dioxan added					
	0 %	5 %	10 %	20 %	30 %	40 %
6·5	2·193	3·906	8·772	11·63	22·73	62·50
6·0	1·972	3·333	6·897	10·0	19·23	52·63
5·5	1·637	2·817	5·291	8·197	15·87	41·67
5·0	1·364	2·392	4·405	6·757	12·50	33·33
4·5	1·100	1·742	3·333	5·618	10·0	27·78
4·0	0·8929	1·486	2·793	4·274	8·333	22·22
3·5	0·7246	1·233	2·222	3·472	6·944	18·52
3·0	0·5556	1·101	1·901	3·012	5·618	15·62
2·5	0·4386	0·8772	1·585	2·500	4·975	11·90
2·0	0·3690	0·6944	1·233	1·972	3·968	10·0

TABLE 5

Sol sample B_3

0·5N KCl (ml.)	$1/t \times 10^4$ in Sec.					
	Amount of Dioxan added					
	0 %	5 %	10 %	20 %	30 %	40 %
6·5	3·333	5·587	7·407	15·87	31·25	58·82
6·0	2·793	4·762	6·250	12·90	25·0	52·63
5·5	2·193	3·906	4·902	10·99	21·28	45·45
5·0	1·862	3·257	4·237	8·772	16·67	37·04
4·5	1·585	2·660	3·472	7·042	14·08	31·25
4·0	1·321	2·169	2·778	5·882	11·49	27·03
3·5	1·101	1·759	2·294	5·00	10·0	21·28
3·0	0·8772	1·399	1·957	3·922	7·813	17·54
2·5	0·7463	1·143	1·661	3·311	6·494	14·08
2·0	0·6061	1·000	1·383	2·801	5·556	11·63

TABLE 6

Sol sample B_3

0·1N KCl (ml.)	$1/t \times 10^4$ in Sec. Amount of Dioxan added					
	0 %	5 %	10 %	20 %	30 %	40 %
6·5	7·143	10·00	18·52	40·0	83·33	250·0
6·0	5·556	8·475	15·62	33·33	71·43	200·0
5·5	4·587	7·042	12·35	27·78	58·82	153·8
5·0	3·906	6·024	10·26	21·74	50·0	125·0
4·5	3·311	5·00	8·547	17·86	40·0	100·0
4·0	2·710	3·922	6·944	16·67	33·33	83·33
3·5	2·190	3·311	5·495	12·35	26·67	66·67
3·0	1·748	2·770	4·587	10·26	22·22	55·56
2·5	1·453	2·398	3·966	8·772	18·52	47·62
2·0	1·233	1·972	3·472	7·937	15·62	40·00

TABLE 7

Sol sample C_1

(Purity = 5·6003)

2·0N KCl (ml.)	$1/t \times 10^4$ in Sec. Amount of Dioxan added					
	0 %	5 %	10 %	20 %	30 %	40 %
6·5	16·78	9·709	12·90	19·53	33·33	69·44
6·0	13·33	7·813	10·64	16·23	26·32	53·19
5·5	10·87	6·536	8·621	12·90	21·98	41·67
5·0	8·13	5·319	6·993	10·64	17·70	34·48
4·5	7·246	4·464	5·714	8·850	14·71	28·17
4·0	5·848	3·597	4·630	6·897	11·90	22·42
3·5	4·762	2·924	3·636	5·882	9·524	20·00
3·0	3·802	2·398	2·924	4·762	7·463	14·93
2·5	3·012	1·946	2·398	3·846	6·024	11·90
2·0	2·506	1·562	1·972	3·125	5·000	10·00

TABLE 8

Sol sample C_2

(Purity = 17·0644)

0·2N KCl (ml.)	$1/t \times 10^4$ in Sec. Amount of Dioxan added 0 %	5 %	10 %	20 %	30 %	40 %
6·5	22·22	14·29	14·93	28·17	37·74	62·50
6·0	17·54	10·99	11·90	22·22	29·85	50·0
5·5	13·99	8·85	10·00	17·99	25·00	40·82
5·0	10·87	7·246	8·333	14·39	21·28	33·33
4·5	8·772	5·495	6·536	11·76	16·67	27·03
4·0	6·667	4·386	5·495	9·524	14·08	20·0
3·5	5·618	3·401	4·464	7·407	11·17	18·52
3·0	4·651	2·817	3·597	6·024	9·347	14·93
2·5	3·788	2·381	2·941	4·808	7·519	11·76
2·0	3·155	1·976	2·506	3·968	6·250	10·00

TABLE 9

Sol sample C_3

(Purity = 34·1281)

0·1 KCl (ml.)	$1/t \times 10^4$ in Sec. Amount of Dioxan added 0 %	5 %	10 %	20 %	30 %	40 %
6·5	27·03	12·90	15·62	17·86	29·41	71·43
6·0	22·22	10·65	12·82	14·08	23·26	55·56
5·5	21·57	8·696	10·87	11·43	20·00	46 51
5·0	13·89	7·042	8·772	9·524	16·67	38·46
4·5	11·43	5·714	6·944	7·937	13·33	29·85
4·0	9·259	4·762	5·882	6·667	10·87	22·73
3·5	7·519	4·000	4·762	5·556	8·850	18·52
3·0	6·024	3·077	3·788	4·651	7·463	14·81
2·5	4·762	1·984	3·067	3·788	6·024	12·05
2·0	3·968	1·821	2·500	3·125	5·00	10·00

The coagulation value for infinite time may be taken as a measure of the stability of a sol. Same has been obtained by extrapolation and are recorded in the following table.

TABLE 10

Values of coagulation in ml of KCl for infinite time at 30°C for sol samples.

% of Dioxan used	A_1	A_2	A_3	B_1	B_2	B_3	C_1	C_2	C_3
	Strength of potassium chloride required								
	2·0N	0·2N	0·1N	2·0N	0·5N	0·1N	2·0N	0·2N	0·1N
0	1·20	1·35	1·40	1·45	1·40	1·50	1·10	1·25	1·35
5	1·05	1·20	1·30	1·25	1·30	1·35	1·25	1·35	1·50
10	0·90	1·00	1·10	1·00	1 05	1·20	1·15	1·30	1·40
20	0·80	0 85	0·90	0·85	0·80	1·05	1·05	1·20	1·20
30	0·70	0·75	0·80	0·65	0·60	0·95	0·80	1·00	1·00
40	0·50	0·60	0·65	0·50	0·45	0·80	0·60	0·80	1·85

TABLE 11

Values of dielectric constant of water and dioxan mixture at 25°C.

Dioxan in ml.	Distilled water in ml.	% of Dioxan in Mixture	Values of dielectric constant
0·0	15·0	0 %	78·5400
0·75	14·25	5 %	72·8448
1·50	13·50	10%	69·9200
3·0	12·0	20%	61·3368
4·5	10·5	30%	51·3473
6·0	9·0	40%	45·3900
15·0	0·0	100%	2·2090

RESULTS AND DISCUSSIONS

It is evident from tables 1-9 that with the lowering of the dielectric constant of the medium, the sols of the hydrous ferric oxide prepared by peptising the same with hydrochloric and with acetic acid became unstable. In the case of the sol obtained by Kreck's method there is first a stabilisation upto 10% of dioxan used and then there is a regular sensitisation, when the dielectric constant is further decreased. The extent of sensitisation or stabilisation increases with the increasing concentration of the added nonelectrolyte. The effect of dioxan is rather more pronounced, when the time of coagulation is small than that observed when the time of coagulation is large as brought about by adding smaller amounts of coagulating electrolyte. Again the degree of sensitisation or stabilisation is more remarkable with the sols having greater amounts of peptising electrolyte *i.e.* with one having lower pH values than one with higher pH. Thus the sols of greater purity shows comparatively lesser sensitivity towards the action of non-electrolyte, dioxan.

The very nature and the behaviour of the various sols prepared by different methods differ remarkably in various respects due to their varying internal structures as we have reported earlier.[7] It is evident here that the sols obtained by Kreck's method behave differently than those prepared by peptising the hydrous oxide by an acid.

It may be pictured that with the decrease in the dielectric constant of the medium, the double layer around the colloidal units becomes mobile and there may be self neutralisation of the charge of the particle leading to the unstability. The surfaces forces operating in aggregation process is also of importance in defining the stability of a sol, which is different for sol particles obtained by different methods. The structure of the colloidal units and the surface of the different sol samples is obviously not the same as the coagula of hydrous ferric oxide sol precipitated from Kreck's sol is more compact than those obtained from other sol samples, where the coagula are highly flocky and have a tendency to form a gel like structure.

Further work is in progress in these laboratories in this direction and will be reported in subsequent publications.

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OPTIMUM USE OF IRRIGATION WATERS IN UTTAR PRADESH

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ABSTRACT

The need for irrigation in a state like Uttar Pradesh and the importance for making the optimum use of irrigation waters therein has been explained. The latter has been advocated by the correct determination of water requirements of crops. An account has been given regarding the experimental studies conducted for three principal crops of Uttar Pradesh. As water requirements vary from one region to another and from crop to crop, the initiation of intensive research by field studies, which can be checked up with formulae given by the United States Department of Agriculture, is recommended not only for a better utilization of irrigation waters but also for the maximization of food production which is the burning problem of the country.

INTRODUCTION

Irrigation waters are used for fulfilling supplemental needs for crop production not met with by natural precipitation. While underirrigation brings down the crop yield, it may not be so well known that over-irrigation also affects agricultural production adversely. Wherever water is available for irrigation to the cultivator, he invariably over-irrigates his fields under the mistaken notion that the higher the application of water for irrigation the greater the yield. This practice is causing a double loss by reducing the yield (besides affecting the quality of the crop) and also by the waste of valuable water for irrigation which could have been utilized for irrigating other fields and thus raising the yield.

In view of the chronic food deficit and limited water resources for irrigation in the country, it is imperative that every drop of water should be put to best use for crop production. Prerequisites for the optimum use of irrigation waters will be a knowledge of the correct water requirements of crops grown in any area and dissemination of the same among the cultivators. With this object in view, a series of experiments were laid out for the determination of water requirements of three principal crops of Uttar Pradesh, namely, wheat, rice and suaar cane at a number of experimental stations.

It may be mentioned that the maximization of agricultural production depends upon proper land and water management. The present series of experiments undertaken in Uttar Pradesh were related to proper water management alone, while regarding the aspect of land management local cultural practices were adhered to as far as possible. For proper management of water which was the real factor experimented upon, the object of the experiments was to work out the best combination of the level of irrigation (or the depth of irrigation treatment) and the frequency of irrigation (interval between irrigation treatments) for obtaining the maximum yield.

The aspect of singling out the factor of proper water management had not been touched upon in previous experiments on the subject in the country. In Uttar Pradesh, some experiments were carried out previously at the Sugarcane Research Station, Shahjahanpur and Rice Research Station, Nagina for the

determination of water requirements of sugarcane and rice respectively, but the watering needs were studied at these stations along with varietal trials and manurial responses. Similar experiments had also been undertaken at Padegaon (Poona) for wheat, Cuttack and Hyderabad for rice. But, as mentioned earlier, no attempts had been made therein to single out the aspect of watering alone.

FUNCTION OF IRRIGATION WATER

The ideal in irrigation is to keep a uniform soil moisture condition during the growing period. However, a water supply that will support this ideal is almost never attained because the period of ample supply is seldom coinc dent with that of greater demand by plants.

Where water is applied for irrigation, the soil in the proximity of the water is immediately saturated, and the gravity portion of the water seeks lower levels under the influence of both gravity and capillary attraction. The wetted soil retains a sufficient quantity of moisture to satisfy its capillary capacity, and all the surplus continues to sink, filling the soil to the capillary capacity as it goes until ground water is reached, when the surplus is consumed in augmenting the ground water table or drains away in whatever channels are available.

All water that is in excess of the optimum water supply and that escapes to the ground water table or passes off as drainage is usually wasted, for it carries mith it in solution plant food. This also results in the rise of the ground water table with its accompanying problems of waterlogging.

At the end of each irrigation, the upper zone of the soil is saturated to a consinerable depth, and the gravity water in this zone slowly descends to lower levels. The moisture of the surface soil soon gets reduced and an upward movement begins of capillary water to make up for the losses from evaporation from the surface. Thus, after each irrigation, a condition is soon reached, in which a narrow zone a short distance below the surface is in a state of capillary saturation, and the percentage of moisture diminishes both upwards and downwards, at a nearly uniform rate. If at this time the moisture at the lower limit of the root zone is just a little above the wilting coefficient, the water has been economically applied, and none wasted. It is therefore imperative that irrigation should be applied before the soil moisture reaches the wilting point throughout the root zone, otherwise, the plants will be severely injured or killed.

It is thus clear that under-irrigation does not fulfil the water requirements of crops and thereby damages the production of the latter. On the other hand, over-irrigation means the application of water in excess of the watering needs. The latter is thus merely a waste of water for irrigation, apart from the fact that the yield and quality of the crop are damaged. This waste of water has other harmful effects as well. The water table keeps on rising steadily owing to the percolation of the excessive quantity of water, which in the course of some years may result in waterlogging with its accompanying disasters in the form of the creation of swamps, damage to public health and sanitation by the spread of malaria, salt encrustation with resulting loss in soil fertility.

It may therefore be repeated that in any agricultural planning, it is essential that both under and over-irrigation should be averted, which is possible only by the optimum use of irrigation waters by the determination of correct water requirements of crops.

EXPERIMENTAL APPROACH

With the objectives set forth earlier, experiments on water requirements of wheat, rice and sugarcane were laid out since 1942-43. The factors studied in these experiments were the level and frequency of irrigation, while other factors such as type of seed, type and quantity of manure and other cultural practices were made to conform to local practices as far as practicable. The experiments were taken up at a number of stations which were typical of varying climatological and topographical features of the State of Uttar Pradesh, namely, Bara Banki, Bulandshahr, Nagina, Tisuhi (Mirzapur), Atarra (Banda), Bahadrabad, Dhanauri. They were replicated over a number of years in order to do away with the effects of climatic variations.

The experiments were conducted with parallel yield trials of different irrigation treatment combinations of level and frequency of irrigation, based on randomized block system. In this method, the experimental area was divided into a number of sub-plots on each of which different dosages and frequencies of irrigation were applied in accordance with a set design. Randomizations in the designed experiments were altered from station to station and from year to year for each crop. The size of sub-plots was governed by the area available for experimentation at each station. Experimental area ranged between 2 and 4 acres at different stations. Subplot sizes were 80′ by 16·5′, 40′ by 22′ or 75′by 15′, 25′ by 11′, 25′by 11′, 25′ by 11′,73′ by 15′,73′by 17.33′ for Barabanki, Bulandshahr, Tisuni, Atarra, Nagina Bahadrabad and Dhanauri respectively. Type of soil was loam at Barabanki and Bulandshahr, black cotton or hard clay soil at Tisuhi, parwa at Atarra, clayey at Nagina, loam mixed with sand and clay at Bahadrabad and Dhanauri.

Particular care was taken to achieve the maximum accuracy in the measurement of depths of irrigation. The discharge at the head of the irrigating gul was usually measured by a 90° V-notch of standard specifications, as detailed in the pamphlet B. S. S. 722·1937. The discharge over a 90° V-notch is calculated by the following equation:—

$$Q = 2{\cdot}49\ H^{2{\cdot}48},$$

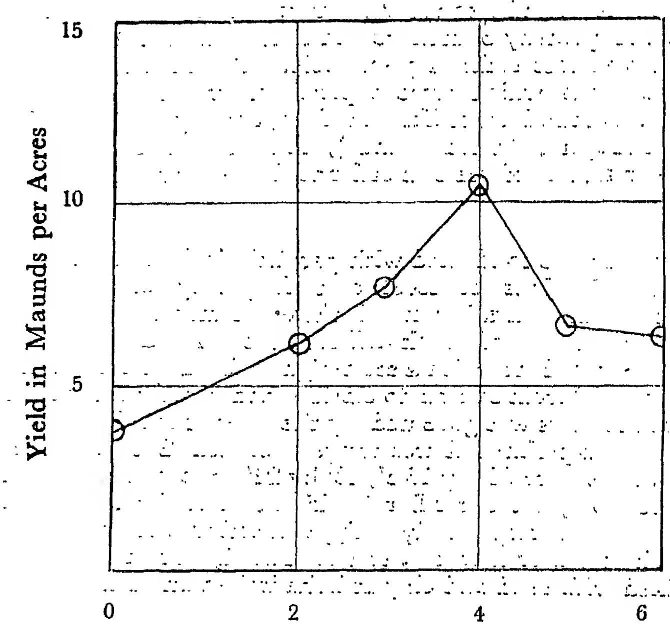
where Q = discharge in cubic feet per second,

H = observed head in feet.

EXPERIMENTAL RESULTS

Yield data for each station was analyzed for each crop every year. A pooled analysis was also carried out at the end of each series of experiments, and the best comoination of irrigation treatments for obtaining the maximum yield was thereby derived for each crop for the different experimental station:

For one irrigation treatment the best yield was obtained with three inches for BaraBanki, Bulandshahr, while this depth was found to be four inches for Atarra.



Graph No. 1. Showing Yield of Wheat at Atarra for one Irrigation Treatment for 1947-48.

From Graph No. 1 for the yield of wheat of Atarra for 1947-48, it will be clear that the yield goes on increasing with increasing depths up to 4 inches above which the yield again goes down, which brings out clearly the harmful effects of both under and over-irrigation.

With two irrigation treatments, the optimum depths were found to be the same with an interval of three to four weeks. Table No. 1 gives an idea of the yield data for Dhanauri for the year 1957-58.

With the limited range of experiments, it has not been possible to study the water requirements for various types of soils prevailing in Uttar Pradesh, although it has generally been found that the requirements were higher for sandy soils as compared to clayey soils, indicating that the soil-moisture retentive capacity does play a role in the watering needs. This is in conformity with the findings of similar experiments carried out in detail by the United States Department of Agriculture. It was also found that the climate or the humidity did not have any pronounced effect on the water requirements of a particular crop, although the first two were the governing factors in the selection of the cropping pattern in any area. It is, however, the rainfall which has a significant effect on water requirements. This has been elucidated below to some extent.

It was brought out that in the drier tracts of Western Uttar Pradesh, represented by Bulandshahr with an annual rainfall of 26 inches, it was essential to have two irrigation treatments, whereas it was not of such primary importance in a wetter place such as Bara Banki with a rainfall of 36 inches. The average percentage ration of the maximum yield due to two and one irrigation treatments was 111 for Bara Banki, whereas this percentage was 147 for Bulandshahr.

A higher optimum depth of irrigation requirement for wheat in Southern Uttar Pradesh (four inches against three inches for Eastern and Western Uttar Pradesh) could only be explained by the important effect of the distribution of rainfall, winter rainfall being practically negligible in Southern Uttar Pradesh unlike that in Eastern and Western Uttar Pradesh. It was also evident that it was not the total quantity of rainfall which mattered most but rather the distribution or pattern of rainfall, for the water requirements for Bara Banki with an annual rainfall of 36 inches were not less for wheat than those for Bulandshahr with an annual rainfall of 26 inches. This could only be explained by the fact that winter rainfall during the growing season of wheat was equally favourable at both the places. It may be mentioned here that in some places of Western Australia it has been possible to grow wheat with an annual rainfall of 10 inches only because of a favourable distribution of rainfall during the growing season.

The effect of water table conditions was not detectable on water requirements of wheat. This factor has not been mentioned in any of the studies conducted by the United States Department of Agriculture.

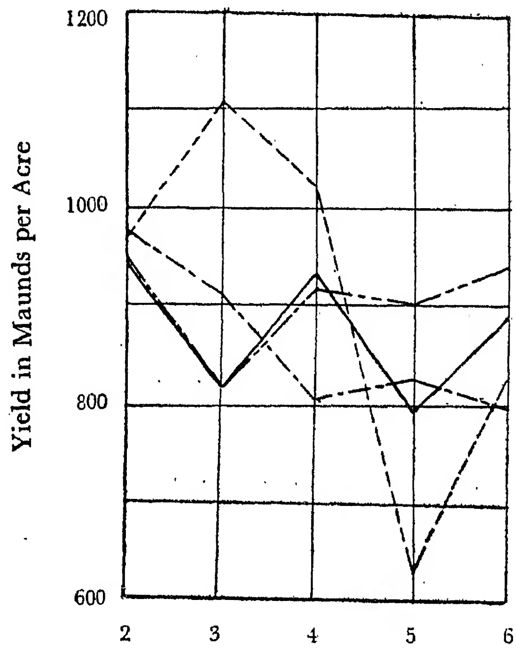
(B) RICE

For this important crop, for which experiments were initiated only for Tisuhi and Atarra and could not be continued over a number of years, a pooled analysis for Atarra for the years 1947 to 1950 indicated the optimum depth of irrigation to be nine inches, no conclusive results being obtained regarding the frequency or interval between irrigation treatments.

(C) SUGAR CANE

The experiments were carried out at Bara Banki and Bulandshahr. Four inches' depth at three weeks gave the best yield for Bulandshahr, while the maximum yield was obtained with five inches depth at intervals of two to three

weeks for Bara Banki. Graph No. 2 for the yield of sugar cane at Bulandshahr for 1948-49 shows clearly how for the best frequency of 3 weeks, the yield is considerably reduced not only by reducing the depth of irrigation but also by increasing the same from 4 inches up to 6 inches. Appendix 1 gives an idea of the analysis of variance of the pooled data for Bulandshahr for the years 1946-47 to 1947-48.



Graph No. 2. Showing Yield of Sugar Cane Co 421 for 1948-49 at Bulandshahr

The results were interesting in that the irrigation requirements for Bara Banki which is actually wetter with 43 inches of rainfall as compared to 28 inches for Bulandshahr were found to be higher, indicating again that the total quantity of rainfall was not the most dominant factor, the pattern of rainfall playing a more important role.

EMPIRICAL APPROACH

Apart from the experimental approach indicated above, an empirical approach for an assessment of the water requirements from climatological date, as advocated by H. F. Blaney of the United States Department of Agriculture, was also attempted for wheat for Dhanauri for the years 1955-56 to 1957-58, vide Table No. 2, which is self-explanatory. The months of November to February, representing the growing season of the (wheat) crop, were taken into consideration. The value of K, the constant in the equation $U = KF$, was assumed from the experimental

data of the United States Department of Agriculture as 0·70 which is recommended for grains with a growing season of 4 to 5 months, while field irrigation efficiency, E in the formula $I = (U\text{-}R)/E$, was assumed to be 70%. Irrigation requirement at the head of the field was thus found to be 10·49 inches. From actual experimental results, it was found that two irrigation treatments of 3 inches each gave the maximum yield. If to this be added a paleo (pre-sowing) irrigation of 3 inches, the irrigation requirements work out to 9 inches which is not much different from that obtained by Blaney's approach, the more so if the seepage losses are also taken into consideration.

CONCLUSIONS

The need for the correct dosage of irrigation for the maximization of crop production has been emphasized. The experimental approach is recommended for typical stations representative of various climatological and topographical classifications. Where this is not possible, the empirical approach of H. F. Blaney may be utilized for an assessment of water requirements from climatological data.

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APPENDIX No. 1

Pooled Analysis of Yield Data of Sugar Cane Experiments conducted at Agricultural School Farm, Bulandshahr during the years 1946-47 and 1947-48 with 3, 4, 5, and 6 inches of irrigation at intervals of 2, 3, 4, 5, and 6 weeks.

The final analysis of variance of Yield of sugar cane of pooled data in $(\text{Seers})^2$ is given in Table I, in which items significant at 5% and 0·1% are marked with one and three asterisks respectively, 40 *seers* being equivalent to 82 lb. The significance of intervals has been tested by years × intervals; that of depths with years × depths, and the interaction of intervals and depths by years × depths × intervals.

TABLE I

Pooled Analysis of Variance of yield of Sugar Cane in $(\text{Seers})^2$

Variation due to	Degrees of freedom	Sum of squares	Mean. square.
Years	1	2,979,270	2,979,270
Blocks	4	25,724,1666	6,431.0417
Intervals (Weeks)	4	24,994,2167	6,248·5542
Depths	3	65,109·0000	21,703·0000*
Week × Depths	12	92,750·9166	7,729·2430***
Years × Weeks.	4	5,447·8833	1,361·9708
Years × Depths.	3	6,807·1334	2,269·0445
Years × Weeks × Depths	12	2,603·7834	216·9820
Error	76	42,281·1677	556·3311
Total	119	3,244,989.	

The following inferences may be drawn from the above pooled analysis ;—

Weeks :—Testing this with years × weeks, we find it to be insignificant.

Depths :—Testing this with years × depths, we see that it is significant at 5% level of probability. Now we form the critical difference for depths based on 30 sub-plots.

$$\text{Critical difference} = \sqrt{\frac{2269{\cdot}0445 \times 2}{30}} \times t_{5\%}\ (3)$$

TABLE I. 1

Summary of Results of Mean Yield in descending order with regard to Depths in the pooled Data.

	4″	5″	3″	6″	c. d. at 5% level of probability
Mean yield in seers per sub-plot	401·2	370·9	359·8	336·4	39·11 seers

Comparing this critical difference (39·11) against the mean yields due to effects of different depths, we find no signficant differences between the mean yields due to four and five inches of irrigation.

Depths × *Weeks* :—The variation due to the interaction of depths and weeks, when compared with the same due to years × weeks × depths, is highly significant. Forming the critical difference at 5% level of probability, we get :—

$$\text{Critical difference} = \frac{\sqrt{216{\cdot}9820 \times 2}}{6} \times t_{5\%}$$

$$= 18{\cdot}52 \textit{ seers} \text{ per sub-plot}$$

TABLE I. 2

Summary of results of Mean yields due to the Effects of Interaction of Depths × Weeks in the Pooled Data.

Weeks / Depths	3″	4″	5″	6″	c. d. at 5%
2. Weeks	356	406·7	428·3	325·8	
3. Weeks	356·3	451·1	360·5	307	
4. Weeks	359	439·2	365·5	374	18·25 seers
5. Weeks	356·2	377·3	367·2	335·2	per sub-plot
6. Weeks	371·8	331·5	333·3	360	

Testing the critical difference (18·52) in the above table against the mean yields due to the effect of this interaction, we see that there are no signficant differences between the mean yields due to 4 inches of irrigation at intervals of 3 and 4 inches. Both are eqaully effective.

CONCLUSIONS

Four inches of irrigation, given at an interval of 3 weeks, may be considered to be the best combination in producing the maximum yield of sugar cane followed by the same depth at an interval of four weeks for optimum yield.

TABLE 1

Showing yield of wheat crop (Pb-591) at Government Agriculture Farm, Dhanauri for the year 1957-58

No. of treatments	Main plot & depth of irrigation	Sub-plot No.	Block I		Block II		Block III		Total		Average yield per sub-plot	Average yield in maunds per acre	Actual dates of irrigation	
			Sr.	Ch.	Sr.	Ch.	Sr.	Ch.	Sr.	Ch.	Seers		Ist irrigation	II irrigation
1	2 ins. (A)	1	6	7	9	0	9	0	24	7	8.125	18.06	6-12-57	6-1-58
2	,,	2	6	0	7	12	9	11	23	7	7.813	17.33	6-12-57	10-1-58
3	,,	3	6	11	9	10	10	8	26	13	8.938	19.82	6-12-57	17-1-58
4	,,	4	7	9	7	14	9	2	24	2	8 188	18.16	6-1-58	3-2-58
5	,,	5	5	5	8	10	9	10	23	9	7.812	17.42	6-1-58	10-2-58
6	,,	6	7	4	8	8	9	9	25	5	8.438	18.71	6-1-58	17-2-58
7	,,	7	6	7	8	8	8	13	25	12	7.875	17.56	17-1-58	14-2-58
8	,,	8	7	8	8	10	9	10	25	12	8.563	19.03	17-1-58	21-2-58
9	,,	9	5	22	2	12	8	12	24	4	4.063	17.93	17-1-58	28-2-58
10	,,	10	7	12	9	8	11	3	28	7	9.500	21.02	24-1-58	21-2-58
11	,,	11	6	1	8	8	10	4	24	13	8.250	18.34	24-1-58	28-2-58
12	,,	12	5	12	9	8	10	0	25	4	8.438	18.66	24-1-58	7-3-58
13	3 ins. (B)	1	5	12	7	5	9	5	22	6	7.438	16.54	6-12-57	6-1-58
14	,,	2	6	10	6	0	10	10	23	4	7.750	17.19	6-12-57	10-1-58
15	..	3	8	0	8	5	6	6	22	11	7.563	15.77	6-12-57	17-1-58
16	..	4	8	2	6	5	9	13	24	4	8.063	17.93	6-1-58	3-2-58

17	,,	5	5	8	6	12	8	4	20	8	6.813	15.15	6-1-58	10-2-58
18	,,	6	7	12	7	15	10	8	26	3	8.750	19.36	6-1-58	17-2-58
19	,,	7	8	5	7	10	10	2	26	1	8.688	19.27	17-1-58	14-2-58
20	,,	8	7	0	7	8	9	0	23	6	7.813	17.37	17-1-58	21-2-58
21	,,	9	6	8	7	10	8	12	22	14	7.625	16.91	17-1-58	28-2-58
22	,,	10	8	2	7	6	8	6	23	14	7.938	17.65	24-1-58	21-2-58
23	,,	11	7	10	9	14	10	7	27	15	9.313	20.65	24-1-58	28-2-58
24	,,	12	7	10	7	9	9	5	24	8	8.188	18.11	24-1-58	7-3-58
25	4 ins. (C)	1	7	12	5	7	5	6	19	9	6.500	14.46	5-12-57	6-1-58
26	,,	2	5	8	6	11	7	12	19	15	6.625	14.74	6-12-57	10-1-58
27	,,	3	8	4	8	12	7	12	24	15	8.313	18.43	6-12-57	17-1-58
28	,,	4	7	14	10	4	8	1	26	3	8.750	19.36	6-1-58	3-2-58
29	,,	5	8	11	7	13	3	9	25	1	8.375	18.53	6-1-58	10-2-58
30	,,	6	7	10	10	1	2	15	27	10	9.188	20.42	6-1-58	17-2-58
31	,,	7	7	1	10	1	7	8	24	10	8.188	18.20	17-1-58	14-2-58
32	,,	8	7	4	9	2	6	13	23	3	7.750	17.14	17-1-58	21-2-58
33	,,	9	8	12	9	2	7	0	24	14	8.313	18.39	17-1-58	28-2-58
34	,,	10	9	4	9	7	8	3	26	14	8.938	19.87	24-1-58	21-2-58
35	,,	11	7	7	9	12	7	0	24	3	8.063	17.88	24-1-58	28-2-58
36	,,	12	7	4	9	2	11	1	27	7	9.125	20.18	24-1-58	7-3-58

Notes :—1. Dates of sowing 6-11-57 to 9-11-57.

2. Dates of harvesting :—15-4-58 to 20-4-58.

3. Rainfall during the crop season:—5.45 inches.

TABLE No. 2

Computations of seasonal consumptive use and irrigation requirements for wheat at the Government Agriculture Farm, Dhanauri, Uttar Pradesh, India on the basis of monthly figures for climatological data from 1955-56 to 1957-58

Month	Mean temperature (t)°F	Per cent day time hours (p)	Consumptive use factor (f)	Average rainfall (r)	Remarks
November ...	66	7.19	4.75	0.41	Per cent daytime hour figures have been computed from Sunshine Table, U. S. Weather Bureau Bulletin 805, 1905 edition for a latitude of 30° for Dhanauri.
December ...	57	7.14	4.07	0.91	
January ...	56	7.30	4.09	2.90	
February ...	59	7.03	4.15	0.38	
			(F)	(R)	
	Total	...	17.06	4.60	

f = monthly consumptive use by wheat = $\frac{t \times p}{100}$.

K = 0.70 = consumptive-use coefficient for wheat adopted from tables given by H. F. Blaney for grain crops with a growing season of 4 to 5 months.

E = field irrigation efficiency which for medium loam is taken to be 70%.

U = KF = consumptive use for growing or irrigation season (November to February in the case of wheat).

F = sum of monthly consumptive use factors, (f) for growing or irrigation season.

I = (U − R)/E = irrigation requirement at head of the field.

Here F = 17.06 inches, so U = KF = 0.70 × 17.06 = 11.94 inches.

I = (U − R)/E = (11.94 − 4.60)/0.70 = 10.49 inches.

EFFECT OF INHOMOGENITY ON THE DECAY OF A SHOCK WAVE

By

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ABSTRACT

The Brinkly and Kirkwood theory of propagation of shoek wave has been extended to the case of an inhomogenous medium. The mechanical decay of shock wave of Mach number five has been studied for three different ratios (0·1, 1 and 2) of the scale behind the shock front to the scale height of the atmosphere. The shock wave is found to decay in the first two cases. In the last case, after decaying over a short distance it begins to increase in strength.

INTRODUCTION

Brinkley and Kirkwood (1947), gave a theory of propagation of a shock wave. The chief feature of this theory is the decaying character of the shock wave as it propagates in a homogeneous medium. This theory has now been extended to take into account the variations of density and pressure in the undisturbed region. A similar extension has also been done by Ono et al (1962) and Kogure and Osaki (1962). The decay of a shock wave in an inhomogeneous stellar atmosphere in some special cases has been studied here by integrating the resulting differential equations. A shock wave of initial Mach number five has been assumed to propagate in an inhomogeneous atmosphere. However, the effect of gravitational field has not been taken into account. The variation of the scale height in the atmosphere has also been neglected. Three different cases specified by the ratios 0·1, 1·0 and 2·0 between the scale behind the shock front and the scale height of the atmosphere have been considered. It is shown that in the first two cases the shock wave decays as it proceeds in the atmosphere, but in the last case it starts gaining strength after decaying over a short distance. It is observed that for a shock to decay, according to this theory, in a stellar atmosphere, it must propagate in a region where pressure and density gradients are relatively small, or in other words the scale height of the atmosphere is larger or of the same order as the scale behind the shock front.

BASIC EQUATIONS

Consider a plane, one dimensional shock wave propagating in an inhomogeneous medium. Neglecting the effect of gravitational field, the equations of hydrodynamics, at the shock front, can be written in a convenient form as:

$$\frac{\rho}{\rho_0} \cdot \frac{\partial u}{\partial x} + \frac{1}{\rho C^2} \cdot \frac{\partial p}{\partial t} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \cdot \frac{\partial p}{\partial x} + \frac{1}{\rho_0} \cdot \frac{\partial P_0}{\partial x} = 0, \tag{2}$$

where u is the particle velocity, p the pressure excess over the pressure P_0 of the undisturbed medium at rest, ρ the density at any instante and ρ_0, the density of

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the undistrubed medium. $\dot{x}$ denotes the Lagrangian coordinate of the shock fron and C the velocity of sound. The undisturbed pressure and density are functions of space coordinate x only.

The physical quantities on the two sides of the shock front are related by the Rankine - Hugoniot relations :

$$p = \rho_0 . u.U, \tag{3}$$

$$\rho(U-u) = \rho_0 U, \tag{4}$$

$$\triangle H = \frac{p}{2}\left(\frac{1}{\rho_0} + \frac{1}{\rho} \right), \tag{5}$$

where U is the velocity of the shock front and $\triangle H$ is the specific enthalpy increment experienced by the gas in crossing the front.

Applying the operator,

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{U} . \frac{\partial}{\partial t} \tag{6}$$

to eqn. (3) one gets

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} - \frac{k}{\rho_0 U} . \frac{\partial p}{\partial t} - \frac{k}{\rho_0} . \frac{\partial p}{\partial x}$$

$$= -u \frac{\partial U}{\partial P_0} . \frac{dP_0}{dx} - u \frac{\partial U}{\partial \rho_0} \frac{d\rho_0}{dx} - \frac{uU}{\rho_0} . \frac{d\rho_0}{dx} , \tag{7}$$

where

$$k = 1 - \rho_0 u . \frac{\partial U}{\partial p} . \tag{8}$$

The fourth equation in the Brinkley-Kirkwood (1947) theory is

$$\frac{1}{p} . \frac{\partial p}{\partial t} + \frac{1}{u} . \frac{\partial u}{\partial t} + \frac{p.u.v}{D(x)} = 0, \tag{9}$$

where

$$D(x) = \int_{t_0(x)}^{\infty} p'.u'.dt \tag{10}$$

$$= \int_{x}^{\infty} \rho_0 . h . d\,\xi_0 \tag{11}$$

and h, is the specific enthalpy increment of an element of the fluid on crossing the shock front and thereafter returning to pressure P_0 . ξ is the displacement at time t of the gas element whose undisturbed position is x . v, is a very slowly varying function. The value of v varies between 2/3 and 1. In the case of a homogeneous medium one can assume that v is independent of x, which is equivalent to imposing

similarity restraint. This assumption may not be exactly true in the present case. However, in view of the limited range of its variation one can simplifity the problem and use a constant value. In the present work a value 2/3 is used. It corresponds to the asymptotic behaviour of the decay curve. p' and u' are the excess pressure and the particle velocity respectively behind the shock front.

From eqs. (1), (2), (7) and (9) one can eliminate $\partial u/\partial x$ and $\partial u/\partial t$ and determine $\partial p/\partial x$ and $\partial p/\partial t$. Substituting these in the equation

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{1}{U} \cdot \frac{\partial p}{\partial t}, \tag{12}$$

one gets,

$$\frac{dp}{dx} = \frac{\nu}{D(x)} \cdot \frac{G}{2(k+1)-G} \cdot \frac{p^3}{\rho_0 U^2}$$

$$- \frac{1}{2(k+1)-G} \left[\left\{ (2-G) - \frac{2p}{U} \cdot \frac{\partial U}{\partial P_0} \right\} \frac{dP_0}{dx} - \left\{ \frac{2p}{U} \cdot \frac{\partial U}{\partial \rho_0} + \frac{2p}{\partial \rho_0} \right\} \frac{d\rho_0}{dx} \right] \tag{13}$$

The eqn (10) gives

$$\frac{dD}{dx} = - \rho_0 h. \tag{14}$$

ISOTHERMAL JUMP ACROSS THE SHOCK FRONT

So far eqs. (13) and (14) are exact. For application to actual cases, one has to make some assumption regarding the path of the gas element behind the shock front. As the temperature across the shock front is extremely high, the radiation flux would be very large and fast. As in earlier works (Odgers and Kushwaha, 1959 ; Bhatnagar and Kushwaha, 1961 *a* and *b*) it is assumed that due to this radiation flux the temperature is equalised instantaneously on the two sides of the shock front. After this the gas element returns adiabatically to the undisturbed pressure P_0. Thus the pressure and density across the shock front may be related as follows :

$$\frac{P}{P_0} = \frac{\rho}{\rho_0} = 1 + s, \tag{15}$$

where s, the shock strength is defined by the relation

$$p = P_0 . s. \tag{16}$$

With the help of the Rankine - Hugoniot equations one gets

$$u = \frac{C_0}{\sqrt{\gamma}} \cdot \frac{s}{(s+1)^{\frac{1}{2}}}, \tag{17}$$

$$U = \frac{C_0}{\sqrt{\gamma}} \cdot (s+1)^{\frac{1}{2}}, \tag{18}$$

$$k = 1 - \frac{s}{2(s+1)} , \tag{19}$$

$$G = 1 - \frac{1}{\gamma(s+1)} , \tag{20}$$

and

$$C^2 = C_0^2 = \gamma \cdot \frac{P_0}{\rho_0} . \tag{21}$$

Introducing these in eqn (13), one gets

$$\frac{ds}{dx} = - \frac{\nu}{D(x)} \cdot P_0 \cdot \frac{\gamma s+\gamma-1}{2\gamma s+3\gamma+1} \cdot \frac{s^3}{s+1} + \frac{\gamma s^2+\gamma s}{2\gamma s+3\gamma+1} \cdot \frac{d\ln\rho_0}{dx}$$
$$- \frac{2\gamma s^2+3\gamma s+\gamma+1}{2\gamma s+3\gamma+1} \cdot \frac{d\ln P_0}{dx} . \tag{22}$$

h, can be written as a sum of the specific enthalpy increments $\triangle H$, on crossing the shock front and $\triangle H'$ that on returning adiabatically to the pressure P_0. Then

$$h = \frac{p}{2} \left(\frac{1}{\rho_0} + \frac{1}{\rho} \right) + \frac{\gamma}{\gamma-1} \left(\frac{P_0}{\rho^*_0} - \frac{P}{\rho} \right),$$

where $\rho_0{}^*$ is the final density. As the gas element follows an adiabate after crossing the shock front, therefore

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0{}^*} \right)^\gamma = 1+s , \tag{23}$$

and hence

$$\frac{dD}{dx} = - \rho_0 h$$

$$= - P_0 \cdot \frac{s(s+2)}{2(s+1)} + \frac{\gamma}{\gamma-1} \cdot P_0 \cdot \left[1-(1+s)^{1/\gamma-1} \right] . \tag{24}$$

INITIAL VALUES

To study how the shock strength s, defined by the relation (16), changes as the shock propagates, one has to intrgrate the two equations (22) and (24) numerically for any particular case under consideration. Suppose that a shock wave of Mach number about five starts propagating in an inhomogeneous atmosphere. In equilibrium state, the pressure at any point x is given by

$$P_0 = P_i \, e^{-x/H} \tag{25}$$

where P_i is the value of P_0 at $x = 0$ and H is the scale height of the atmosphere. Strictly speaking H should vary in space, but for the sake of simplicity, as an approximation this variation is neglected. From the definition of the scale height (Lamb, 1959) one can show that if the variation of H is not considered then

$$\frac{d\ln P_0}{dx} = \frac{d\ln \rho_0}{dx} . \tag{26}$$

Introducing the dimensionless quantities, ζ and y defined as follows

$$D(x) = P_0 \cdot H \cdot \zeta , \tag{27}$$

and

$$x = H \cdot y , \tag{28}$$

the eqs. (22) and (24) reduce to

$$\frac{ds}{dy} = -\frac{\nu}{\zeta} \cdot \frac{\gamma s+\gamma-1}{2\gamma s+3\gamma+1} \cdot \frac{s^3}{s+1} + \frac{\gamma s^2+2\gamma s+s+\gamma+1}{2\gamma s+3\gamma+1} , \tag{29}$$

and

$$\frac{d\zeta}{dy} = -\frac{s(s+2)}{2(s+1)} + \frac{\gamma}{\gamma-1}\left[1-(1+s)^{1/\gamma-1}\right] + \zeta . \tag{30}$$

The second term in eq. (29) and the last term in the eqn. (30) are the contributions due to the variations of undisturbed pressure and density. The steeper the gradients of pressure and density, larger is ζ and hence slower is the decay. We can take $\gamma = 5/3$ for a monatomic perfect gas and $\nu = 2/3$. With the help of eqn. (17), initial value of $s = 40$ is taken for a shock wave of Mach number five. The initial value of ζ is found below :

$$\zeta = \frac{1}{P_0 H}\int_{t_0(x)}^{\infty} p' \cdot u' \cdot dt = \frac{l}{H}\int_{t_0(x)}^{\infty} u' \cdot dt = \frac{l\,d}{H} ,$$

where p' has been averaged as $l \cdot P_0$; l being a constant. Following Odgers and Kushwaha (1959) we arbitrarily take $l = 10$. The integral $\int_{t_0(x)}^{\infty} u' \cdot dt = d$ gives the scale behind the shock front or the distance travelled by the material. It can be determined by integrating the observed radial velocity curve for any particular star. We have considered three values of the ratio of d (the scale behind the shock front to H (the scale height) namely, 0·1, 1 and 2. These give the initial values of $\zeta = 1{\cdot}0$ 10·0 and 20·0 respectively.

DISCUSSION OF THE RESULTS

The eqs. (29) and (30) have been integrated numerically following Härm and Schwarzschild Scheme (1955) for the above mentioned three different ratios of the scale behind the shock front to the scale height of the atmosphere. The values of

ζ, s, u/C_0 and U/C_0 obtained in this way as the shock propagates have been given in tables 1, 2 and 3 for the three cases respectively. u/C_0 and U/C_0 have been plotted

TABLE 1

Decay of a Shock wave in an Inhomogeneous Atmosphere—(d/H = 0·1)

y	ζ	s	u/Co	U/Co	$x \times 10^{10}$cm.
0	1·000	40·000	4·84	4·96	0
0·01	0·835	35·324	4·54	4·67	0·50
0·02	0·691	30·995	4·25	4·38	1·00
0·03	0·567	27·007	3·96	4·10	1·50
0·04	0·459	23·357	3·67	3·82	2·00
0·05	0·368	20·039	3·39	3·55	2·50
0·06	0·291	17·048	3·11	3·29	3·00
0·07	0·228	14·373	3·84	3·04	3·50
0·08	0·176	12·007	2·58	2·80	4·00
0·09	0·134	9·936	2·33	2·56	4·50
0·10	0·100	8·147	2·09	2·34	5·00
0·11	0·174	6·624	1·86	2·14	5·50
0·12	0·054	5·344	1·64	1·95	6·00
0·13	0·039	4·286	1·44	1·78	6·50
0·14	0·028	3·424	1·26	1·63	7·00
0·15	0·020	2·731	1·10	1·50	7·50
0·16	0·014	2·179	0·95	1·38	8·00

TABLE 2

Decay of a Shock wave in an Inhomogeneous Atmosphere—($d/H = 1\cdot 0$)

y	ζ	s	u/C_0	U/C_0	$x \times 10^{10}$ cm.
0·00	10·000	40·000	4·84	4·96	0·00
0·20	8·412	34·360	4·48	4·61	1·00
0·40	7·049	29·440	4·14	4·28	2·00
0·60	5·887	25·165	3·81	3·96	3·00
0·80	4·900	21·467	3·51	3·67	4·00
1·00	4·067	18·282	3·23	3·40	5·00
1·20	3·369	15·554	2·96	3·15	6·00
1·40	2·785	13·225	2·72	2·92	7·00
1·60	2·300	11·250	2·49	2·71	8·00
1·80	1·900	9·580	2·28	2·52	9·00
2·00	1·570	8·175	2·09	2·35	10·00
2·20	1·299	6·997	1·92	2·19	11·00
2·40	1·077	6·012	1·76	2·05	12·00
2·80	0·744	4·500	1·49	1·82	14·00
3·20	0·518	3·441	1·27	1·63	16·00
3·60	0·359	2·685	1·08	1·49	18·00
4·00	0·246	2·126	0·93	1·37	20·00

TABLE 3

Computation of the Shock Strength of a Shock wave as it Progagates in an Inhomogeneous Atmoshphore—($d/H = 2{\cdot}0$)

y	ζ	s	u/C	U/Co	$x \times 10^{10}$ cm.
0·00	20·000	40·000	4·84	4·96	0·00
0·2	20·368	39·139	4·79	4·91	0·50
0·4	20·900	38·495	4·75	4·87	1·00
0·6	21·609	38·058	4·72	4·84	1·50
0·8	22·632	37·821	4·70	4·83	2·00
1·0	23·632	37·781	4·70	4·83	2·50
1·2	24·993	37·932	4·71	4·84	3·00
1·4	26·629	38·276	4·73	4·86	3·50
1·6	28·579	38·816	4·77	4·89	4·00
1·8	30·892	39·553	4·81	4·93	4·50
2·0	33·624	40·494	4·87	4·99	5·00
2·4	40·646	43·026	5·03	5·14	6·00
2·8	50·397	46·509	5·23	5·34	7·00

against distance travelled by the shock wave in the figures 1, 2 and 3. It is seen that the shock decays rapidly in the first case, rather slowly in the second, but in

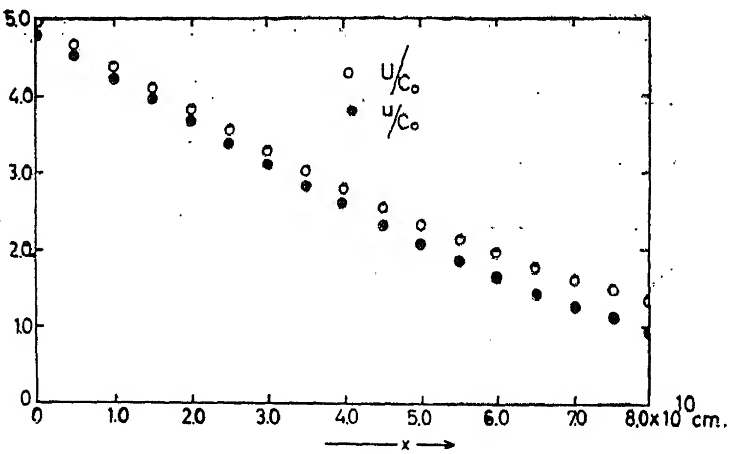

Fig. 1. The theoretically computed velocities have been plotted against the distance for $d/H = 0{\cdot}1$. Open circles give U/Co and dark circles give u/Co.

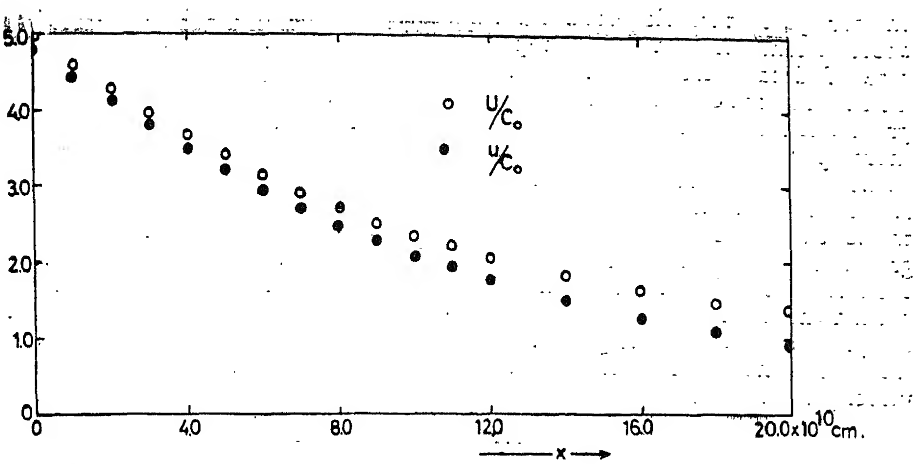

Fig. 2. u/Co and U/Co have been plotted against x, the distance travelled by the shock wave from where it first becomes observable. For $d/H = 1$. Open circles represent U/Co and dark circles represent u/Co.

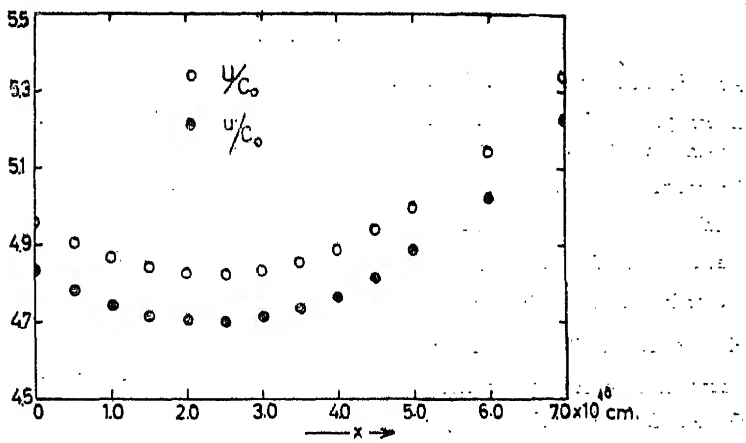

Fig. 3. u/Co and U/Co versus x curve for $d/H = 2$. Open circles represent U/Co and dark circles represent u/Co.

the third case it decays only over a small distance and thereafter starts gaining strength. From this it is obvious that for higher values of these ratios the growth would be still faster. Odgers and Kushwaha (1959) and Bhatnagar and Kushwaha (1961 *a*, and *b* and 1962) have used the Brinkley-Kirkwood theory to compute theoretically the variations of the observed velocities of a β -Cephei star BW Vul treating the atmosphere to be homogeneous. This roughly corresponds to the first case considered here. Usually the scale heights are determined from the gravitational acceleration. Whitaey (1956 *a* and *b*) has shown that the effective value of gravity is reduced due to propagation of a shock wave. Recently Iróshnikov (1962) has shown from analysis of the behavour of H_α , on the basis of a model in which the shock wave progagates in a uniform gravitational field that the approximate value of the effective gravitational acceleration is of the order of 40 cm/sec^2 for RR Iyrae.

In our present work we have not taken into consideration the gravitational acceleration. If we do so perhaps the second case will also give sufficiently rapid rate of decay and assuming the scale height to be given by this case for BW Vul, *i.e.* equal to the distance through which the shock propagates, the effective value of gravitational acceleration for this star will be of the order of 50 cm/sec^2. However there appears to be some uncertainty about the scale behind the shock front. The value adopted here was given by Odgers (1955) which appears to be larger by a factor of about five. Abt (1959) has given the mean total expension for this star to be twice the scale height of the atmosphere (calculating the scale height from gravitational accelerations). If Whitney's reduction factor is applied to Abt's value, then scale height of the atmosphere may be taken to be of the order of the scale behind the shock front. If it could be shown that in the relevant region of the atmosphere such a scale height (of the order of the distance travelled by the material) exists, that the shock model can be usefully employed to explain the observed radial velocity variations.

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NUCLEO-SYNTHESIS DUE TO HELIUM BURNING IN STARS

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ABSTRACT

The rate of production of Ne^{20} from O^{16} $(\alpha,\gamma)Ne^{20}$ reaction through its 5·64 (3−) and 5·80 (1−) Mev excited states is investigated and found to be considerably low in comparison with other helium burning reaction rates in the temperature rage $1-2\times10^{8}$ °K and $\rho=10^5$ g/c.c. On the other hand N^{14} (α,γ) F^{18} $(\beta+\nu)$ O^{18} reaction rate through 4·651 Mev excited level of F^{18} is calculated and found to be faster at these temperatures. This depletion of N^{14} will facilitate heavy element synthesis through neutron capture because N^{14} is a rapid absorber of neutron through N^{14} (n,p) C^{14} reaction.

At temperatures in the range of $3-6\times10^{8}$ °K the rates of formation of Ne^{20} and Mg^{24} are comparatively faster. At still higher temperatures the photodisintegration of Ne^{20} becomes important. The photo dissociation and other heavy-ion reactions leading to the destruction of Ne^{20} are considered. Rates of production of Ne^{22} and Mg^{26} through successive helium captures are considered at these temperatures.

INTRODUCTION

Helium burning reactions start in the core of the red giant stage of stellar evolution after the exhaustion of hydrogen as the energy producing material. The first reaction that takes place in the helium core at $T \sim 10^8$ °K is the triple alpha collision forming C^{12} in its 7·66 Mev excited state. Then subsequent captures of helium by C^{12} produce O^{16} and so on. Hayakawa et al (1956) calculated the reaction rates leading to the formation of C^{12}, O^{16} and Ne^{20} under the condition at temperature $\sim 10^8$ °K. The rates of forming C^{12} and Ne^{20} were found to be larger. Nakagawa et al (1956) made detailed calculations on the helium capturing reactions. Many other authors (Burbidge et al, 1957 ; Salpeter, 1957) studied the fusion of helium as a mode of element synthesis in stars. The helium burning processes were found to be responsible for the synthesis of C^{12}, O^{16}, Ne^{20} and perhaps Mg^{24}.

Recent experimental investigations and results thereof warrant a re-investigation of the reaction rates during the helium burning phase of stellar evolution. The production of Ne^{20} was expected to be due to the 4·97 Mev excited state of Ne^{20} whose spin and parity where supposed to be 2+. The formation of Ne^{20} through O^{16} (α,γ) Ne^{20} reaction requires that this level should have either odd-spin, odd-patity or even-spin, even-parity. But experiments (Litherland et al, 1961 ; Almqvist et al 1961) showed that this level has even-spin and odd-parity and hence might not be formed by α−capturing reaction in O^{16}. There are other two leavels of Ne^{20} in the relevent energy regions of Ne^{20} having odd-spin and odd-parity. Thus it is expected that these two levels will contribute to the resonance formation of Ne^{20} in stellar interiors.

In the present paper attempt is made to examine the rates of formation of helium burning products with the newly acquired experimental informations. Helium bruning reactions at different temperature ranges are considerd. Tempera-

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tures in the range $T=3\times10^{8}$ °K to 6×10^{8} °K and higher may be expected to be prevalent in the late type giants or pre-supernova stages and it is believed that Ne^{20} will be produced at these stages.

It was suggested (Cameron, 1957) that hydrogen from the envelope may be mixed to the expanding helium core in globular cluster star at the tip of the giant branch. Considerable amounts of carbon cycle products would then take part in the helium burning reactions. Neutrons, produced through C^{13} (α, n) O^{16} which begins even at slightly lower temperature ($\sim 8\times10^{7}$ °K) will soon be depleted by N^{14} (n, p) C^{14} reaction and enough neutrons will not be available for heavy element synthesis. To facilitate heavy element synthesis it is required that N^{14} should be used up by N^{14} (α, γ) F^{18} (β^{+}, ν) O^{18} reaction (Burdidge et al, 1957), so that neutron consumption may not take place on a lavish scale. N^{14} (α,γ) F^{18} $(\beta^{+}\nu)$ O^{18} resonant reaction, therefore, is considered in the present note and similarly O^{18} (α,γ) and Ne^{22} (α,γ) non-resonant rates are also studied.

The non-resonant rate of formation of Ne^{22} is quite low and so is the Ne^{22} (α,γ) Mg^{26} reaction. Considering that at higher temperatures ($T_8 > 5$) several resonance levels are present within the 'Gamow peak' in Mg^{24}, the reaction cross-section for Ne^{20} (α,γ) Mg^{24} is obtained as an average over many resonances and the reaction rates are found to be quite faster.

The photo-dissociation rates at higher temperatures for Ne^{20} are calculated. At higher temperatures when heavy-ion collisions can take place the destruction of Ne^{20} through $C^{12}+Ne^{20}$ and $O^{16}+Ne^{20}$ are also considered.

Protons and alpha-particles that are produced in the heavy ion reactions are being utilized for still further heavier element synthesis.

RESONANCE AND NON-RESONANCE REACTIONS

The rate of formation of C^{12}, Ne^{20}, and O^{18} (through F^{18}) are considered to be through the resonant reactions because at relevent energy regions, excited states of these compound nuclei are found to exist.

The resonant rate of formation of the compound nucleus is given by (Reeves and Salpeter, 1959)

$$\mathrm{Log}_{10}{}^{P} = 12{\cdot}69 + \log_{10}\frac{\rho x_\alpha}{A_\alpha}\,\omega\,\frac{\Gamma_\alpha\ \Gamma_\gamma}{\Gamma} - \frac{3}{2}\log_{10}\ (AT_8) - \frac{50{\cdot}4}{T_8}\ E_R\ \mathrm{Sec.}^{-1} \quad (1)$$

where, the widths Γ_α, Γ_γ and Γ and resonance energy E_R are all measured in Mev, and tempetature is in 10^{8} °K units. A is the reduced mass in AMU of the two interacting particles and A_α that of helium nucleus. Γ_α and Γ_γ are respectively the alpha-particle and radiation widths. The total width $\Gamma=\Gamma_\alpha+\Gamma_\gamma$ $\omega=(2J+1)/(2S+1)(2I+1)$, where J is the spin (in the unit of $\hbar$) of the compound state formed and S and I are the spins of the two interacting nuclei.

Wherever the experimental values of $\omega\Gamma_\alpha\Gamma_\gamma/\Gamma$ are known their use has been made in this paper. Otherwise, considering that $\Gamma_\alpha << \Gamma_\gamma \approx \Gamma$, it is taken

that the second term on the right hand side of (1) is reduced to Γ_α. The alpha-particle width Γ_α is obtained by using Wigner and Isenbud's (1947) disperson relation of nuclear reactions which gives

$$\Gamma\alpha = 2K_\alpha \; P_l \, \gamma^2_\alpha , \tag{2}$$

where K_α is the wave number of the incident particle, P_l is the barrier penetration probability for particle of angular momentum l and γ_α^2 is the reduced width of the particle in the level under consideration. The particle width is represented with a subcript α throughout the following because alpha-particle reactions are considered here

$$P_l \approx G_l^{-2} , \text{ (for Low energies)} \tag{3}$$

where G_l is the irregular solution of the Schrödinger wave equation for a particle in a Coulomb field.

At very low bombarding energies the S - wave penetrability is (Cameron, 1959a)

$$P_0 \approx G_0^{-2} \simeq 2\pi\eta\theta_0^2 \exp(-2\pi\eta), \tag{4}$$

where $\eta = 0{\cdot}1574 \, Z_0 \, Z_1 \, A^{\frac{1}{2}} / E^{\frac{1}{2}}$, E is the bombarding energy in C–M system in Mev.

$$\theta_0^2 = x^2 \, K_1^2 (x) \text{ and } x = (8\rho\eta)^{\frac{1}{2}}$$

where $K_1(x)$ is the modified Bessel function (Burbidge et al, 1957) given by

$$K_1(x) \simeq (\pi/2x)^{\frac{1}{2}} \exp(-x + \tfrac{1}{2}x) \tag{5}$$

and $\rho = KR$. R, the interaction radius is taken to be

$$R = 1{\cdot}45 \times 10^{-13} \left(A_0^{\frac{1}{3}} + A_1^{\frac{1}{3}} \right) \text{cm.}$$

γ^2_α is the reduced width given by (Blatt and Weisskopft, 1952),.

$$\gamma^2_\alpha / \bar{D} = 2 \times 10^{-14} \text{cm.} \tag{6}$$

By knowing $\bar{D}$ from the energy level scheme one can determine γ^2_α and thus Γ_α in (2) will be completely known. γ^2_α in (2) is always smaller than $\gamma^2_{\alpha,\max}$ which is called the single particle limit of the reduced width ($\sim h/mR$, m being the reduced mass of the interacting nuclei).

For the non-resonant thermonuclear reaction between particle of type 0 and 1, the number of reactions per nucleus of type 1 is (Burbridge et al, 1957);

$$p_{n,r} = 4{\cdot}34 \times 10^5 \frac{\rho x_0}{A_0} S\,(AZ_0Z_1)^{-1} \tau^2 e^{-\tau} \sec^{-1} \tag{7}$$

In the present case particles of type 0 are helium nuclei. ρ is the density in g/cc ; x_0 is the concentration by weight of alpha-particles ; A is the reduced mass in A M U of the two interacting nuclei and S, the cross-section factor in Kev-barns is given by

$$S = \sigma E \exp\,(\,31{\cdot}28\, Z_0Z_1A^{\frac{1}{2}}\, E^{-\frac{1}{2}}\,)\ \text{Kev, barns,} \tag{8}$$

where σ is the reaction cross-section in barns and E is the particle energy in the CM system in Kev. E can be determined from (Burbidge et al, 1957)

$$E = 1{\cdot}220\,(\,Z_0^2\, Z_1^2\, AT_8^2\,)^{1/3}\ \text{Kev.} \tag{9}$$

Consequently for different kinetic temperatures σ can be determined by the usual methods. τ in (7) is given by

$$\tau = 42{\cdot}48\,(\,Z_0^2\, Z_1^2\, A\,)^{1/3}\, T_8^{-1/3}, \tag{10}$$

where T_8 is the temperature in the unit of $10^{6\,0}$K.

For Mg^{24}, the cross-section is determined as an average over many resonances with the help of expression (Cameron 1959b)

$$< \sigma > = \frac{2\pi^2}{K_\alpha^2} < \omega \frac{\Gamma_\alpha\ \Gamma_\gamma}{\Gamma D} > \times 10^{24}\ \text{barns} \tag{11}$$

where D is the average level distance and the other terms have their usual meanings

With certain assumptions this expression can be written as

$$< \sigma > = \frac{4\pi^2}{K_\alpha} P_0 <\omega> < \gamma^2/D > \times 10^{24}\ \text{barns.} \tag{12}$$

By knowing $< \sigma >$ from (12), one can calculate with the help of (7) and (8) the rate of formation of Mg^{24}. $<\omega>$ is taken to be equal to 3.

For the non-resonant $O^{18}(\alpha, \gamma)Ne^{22}$ and $Ne^{22}(\alpha, \gamma)Mg^{26}$ first the cross-sections for (α, n) reactions are obtained (Cameron 1959a). Since $\Gamma\alpha << \Gamma_n \approx T_1$ $< \sigma > (\alpha,\gamma)/< \sigma >(\alpha, n) \simeq \Gamma_\gamma/\Gamma_n$. We have taken this ratio to be of $\simeq$ 0·01. Thus $< \sigma > (\alpha, \gamma)$ *s* are obtained and with the help of (8) and (7), the reaction rates are calculated.

The rate of photo-dissociation involving any one nuclear resonance is given by (Cameron, 1959a)

$$\lambda_r = 1{\cdot}50\times10^{15}\ \frac{\omega_0\omega_1}{\omega_{01}}\ \gamma\ \exp 11{\cdot}61\ \frac{(Q-E_R)}{T_9}\ \text{sec.,}^{-1} \tag{13}$$

where ω_0, ω_1 and ω_{01} are the statistical weights of the two interacting nucle 0 and 1 and their product respectively and

$$\gamma = \omega\ \Gamma_\alpha\ \Gamma_\gamma\ /\ \Gamma$$

E_R is measured in Mev and widths are in ev. Temperature is in 10^9 °K unit. λ^{-1} gives the mean life-time of photo-disintegration rate for a given energy level.

CALCULATIONS

The helium bruning reaction rates are calculated at different temperatures. Ne^{20} production rate is calculated from the two excited levels separately. The quantity $\omega\ \Gamma_\alpha\ \Gamma_\gamma\ /\ \Gamma$ for the 5·64 Mev and 5·80 Mev levels is calculated by Gove et al (1961) and Kuehner et at (1961) respectively. These values have been used in the present calculations. The quantity for 5·80 Mev level is found to be little less than 0·15 ev and in the present case a value of 0·10 ev is taken and for 5·64 mev level $\omega\ \Gamma_\alpha\ \Gamma_\gamma\ /\ \Gamma = 0{\cdot}003$ ev. For the N^{14} (α,γ) F^{18} resonant reaction rate, the different parameters for the 4·651 Mev excited state are taken from the work of Cameron (1959c). ($\gamma^2_\alpha=0{\cdot}01$, $\gamma^2_{\alpha,\max}= 2{\cdot}3\times10^{-9}$ ev. cm.). It is, however, assumed that this level will be formed due to the reaction under consideration. $E_R = 0{\cdot}23$ Mev. The particle width is calculated by the method outlined above. Γ_α is found to be $4{\cdot}0\times10^{-11}$ ev. Taking $\rho = 10^5$ g/c.c. (pure helium) all the reaction rates at these temperatures are calculated.

The $3\alpha\rightarrow C^{12}$ reaction rates are calculated with a slightly modified value of Γ_γ due to Alburger, (1961) for the second 7·66 Mev excited state of C^{12}. It is assumed that the second excited state decays to the ground state through cascade γ—ray transition, through the emission of 3·23 Mev gamma-radiation. Γ_γ is taken to be $2{\cdot}5\times10^{-3}$ ev with an accuracy of $\sim 50\%$. The equation that is used for this set of calculations is taken from Burbidge et al (1957).

The other source of neon in stars may be due to O^{18} which is produced by α—capturing reaction in N^{14}. The non-resonant reaction rates are calculated by the

methods outlined above. The O^{16} formation rate is taken from the work of Burbidge et al (1957) and shown in Fig. 1.

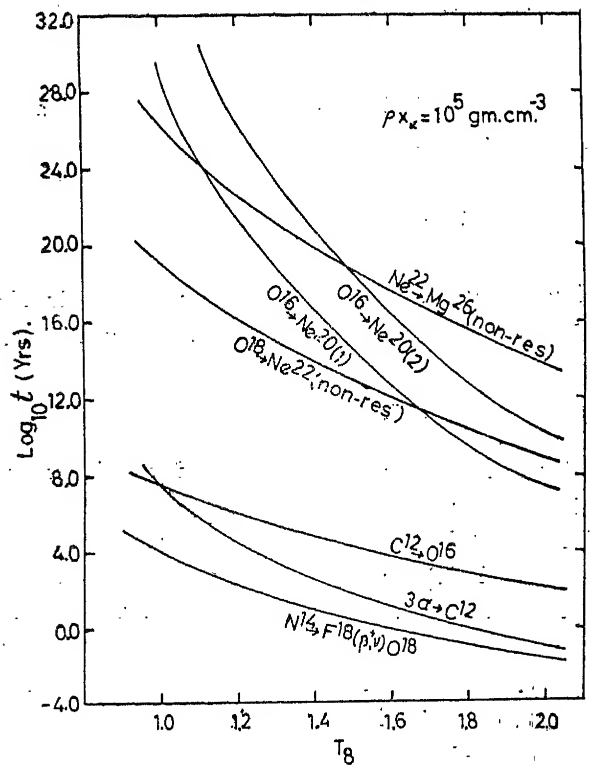

Fig. 1. Helium burning mean life-times vrs temperature. (t is in years and temperature is in 10^8 °K-unit. $\rho x_\alpha = 10^5$g/c.c.) (1) and (2) against $O^{16} \to Ne^{20}$ give the mean life times of this reaction for the 5·64 and 5·80 Mev levels respectively.

The reaction rates are shown in Fig. 1 for the temperature range $1-2 \times 10^8$ °K. The formation rates of Ne^{22}, Mg^{26}, Ne^{20} (from the two levels under consideration, separately) and Mg^{24} in the temperature range $3\text{-}6 \times 10^8$ °K are shown in fig. 2.

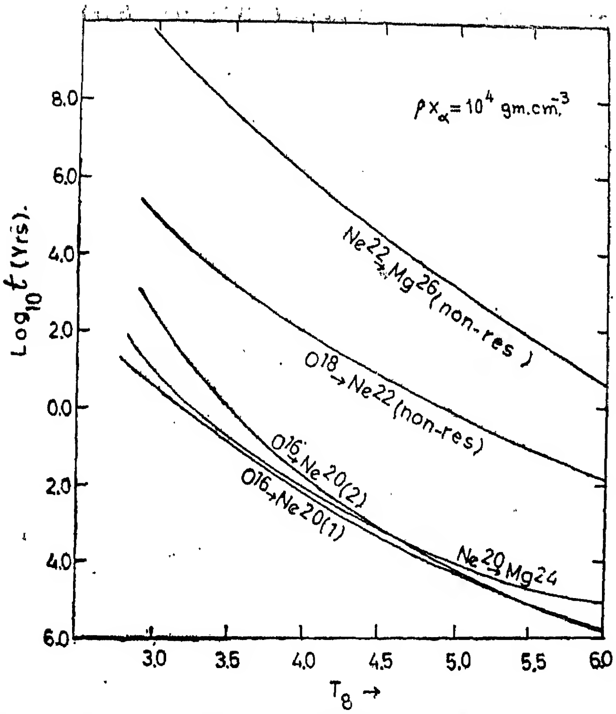

Fig. 2. Helium burning reaction rates as a function of temperature. (1) and (2) against $O^{16}\rightarrow Ne^{22}$ show the rates of 5·64 and 5·80 Mev levels separately.

Table I gives some of the mean life-times of reactions at a few different temperatures.

TABLE I

Logarithm of the mean life-times (t) in helium burning (yrs).

T_9	$3He^4\rightarrow C^{12}$	$C^{12}\rightarrow O^{16}$*	$O^{16}\rightarrow Ne^{20}$ (1)	$O^{16}\rightarrow Ne^{20}$ (2)	$N^{14}\rightarrow F^{18}(\beta,{}^{+}\nu)O^{18}$	$O^{18}\rightarrow Ne^{22}$	$Ne^{22}\rightarrow Mg^{26}$
1·0	7·49	7·57	29·05	36·09	3·92	19·06	26·20
1·2	4·59	5·94	21·78	27·39	2·11	–	–
1·4	2·56	–	16·60	21·19	1·09	13·86	20·10
1·5	1·76	4·09	14·53	18·71	0·32	–	–
1·8	−0·08	–	9·72	12·95	−0·85	10·34	15·99
2·0	−0·98	1·91	7·33	10·08	−1·75	8·95	13·38

*This result is taken from Burbidge et al (1957).

At higher temperatures when γ-rays acquire sufficient energy to disrupt the ground state of nuclei, photo-dissociation becomes a very important phenomenon. Photo-disintegration rate depends on the disintegration energy which is comparatively smaller (4·75 Mev) in case of Ne^{20}. Destruction rate of Ne^{20} is quite considerable at temperature $\sim 10^9$ °K which is shown in Fig. 3.

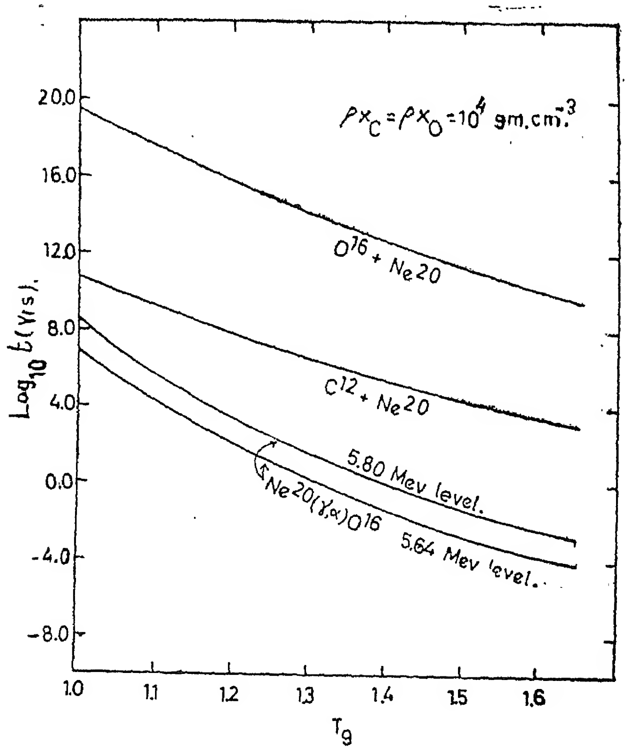

Fig. 3. The photo-disintegration reaction rates vrs temperature. The heavy-ion collision times are also shown against temperature.

Other possible destruction rates of Ne through heavy-ion collisions such as $C^{12}+Ne^{20}$ and $O^{16}+Ne^{20}$ have also been considered. Ne^{20} collision with itself takes place at still higher temperatures and is therefore not considered here.

RESULTS AND DISCUSSION

The helium burging time-scale is supposed to be $\sim 10^7$ years (Burbridge et al, 1957). From the present calculations it seems that during this period negligible amount of O^{16} is converted into Ne^{20} due to helium burning. After a time $\sim 10^7$ years has elapsed the reaction O^{16} (α,γ) Ne^{20} begins at T $\sim 2\times10^8$ °K. Therefore, an equilibrium situation with O^{16} (α,γ) Ne^{20} and Ne^{20} (α,γ) Mg^{24} as constituents cannot be envisaged. Since nothing can be said about the mode of evolution of stars after this stage it is hardly possible to estimate anything about the helium burning reactions leading to the production of Ne^{20}, Mg^{24}, etc. which occur at temperatures higher than 2×10^8 °K.

At higher temperatures, supposed to be prevalent in the core of the late giants or pre-supernova stage of a star, the rate of production of Ne^{20} becomes quite faster. But, since life history of the star at this stage is not known the ratio of abundances is difficult to estimate with any certainity. However, the reaction rate of Ne^{20} (α,γ) Mg^{24} is definitely an over-estimate at lower temperatures. But as helium is exhausted at lower temperatures, further helium burning reactions do not proceed unless a fresh supply of helium is available. At this stage, heavy ion reactions such as C^{12} (C^{12},α) Ne^{20} can provide a helium source; but it takes place at temperature of $\sim 6\times10^{8}\,{}^{o}K$ (Cameron 1959b). Otherwise, the helium burning reactions (fig. 2) may be envisaged to be taking place at the supernova helium burning shell sources where the temperature is considerably higher.

From table I, one can see that the reaction rates at $T_6=120$, for all the reactions $3\alpha \rightarrow C^{12}$, C^{12} (α, γ), are roughly the same. Ne^{20} production rate would, however, be too slow by a factor 10^{16}.

Ne^{22} is formed through O^{18} (α, γ) and it depends on N^{14} (α,γ) $F^{18}(\beta^{+}\nu)$ O^{18} reaction. The binding energy of an alpha-particle in F^{18} is 4.42 Mev and there are excited states of this nuclei at 4·40, 4·651 and 4·74 Mev. If the 4·651 Mev level has the right spin and parity (only the level should not have 0+) then a resonance rate of formation of F^{18} is faster. This F^{18} emits neutrino (half-life $\simeq$1·87 hr) with the release of energy $\sim$0·3 Mev and goes over to O^{18} (Cameron 1959c). This rate is found to be faster than $3\alpha \rightarrow C^{12}$ reaction at the helium burning temperatures (Table I). Therefore it can be expected that this reaction would compete with $3\,\alpha$ reaction as a source of energy generation at these temperatures and density as was also suggested by Cameron (1959c). It may be hoped that Ne^{22} is made also, as was pointed by Cameron (1957), through Ne^{18} (α,γ) Mg^{22} $(\beta^{+}\nu)$ Na^{22} $(\beta^{+}\nu)$ $Ne.^{22}$ Ne^{22}, thus formed, may be destroyed by (p,γ), (α,n), (α,γ) and (n,γ) reactions. Table II shows a cross-section taken from the work of Cameron (1959b). The last two values are calculated according to the usual method. It seems that Ne^{22} synthesis will be considerably low in these steller conditions.

TABLE II*

Reactions with Ne^{22} and their reaction cross-sections calculated at $T_8=6$.

Reactions	$<\sigma>$
$Ne^{22}(p,\gamma)$ Na^{23}	0·22 micro-barns†
Ne^{22} (α,n) Mg^{25}	0·2 Micro-micro barns
Ne^{22} (α,γ) Mg^{26}	0·002 Micro-micro barns

*Since $\Gamma_\gamma << \Gamma_n \approx \Gamma$, $<\sigma>(\alpha,\gamma)/<\sigma>(\alpha,n) \approx \Gamma_\gamma / \Gamma_n$ and this ratio is taken to be equal to 0·01.

†Cameron (1959b).

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DECAY OF INTENSE SHOCK WAVES IN STELLAR ATMOSPHERE

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ABSTRACT

The mechanical decay of a shock wave taking into consideration the interaction of material gas with radiation and including the loss of energy by radiation, has been studied with a view to interpret the radial velocity variations in β -cephei stars. The temperature jump is found to be much less than that for a perfect gas.

INTRODUCTION

Bhatnagar and Kushwaha, in an earlier communication (1961 a), have studied the Brink-ley-Kirkwood theory (1947) of propagation of shock waves for application to interpret the discontinuous radial velocity curve of β-cephei stars, taking into account the interaction of material gas with radiation. Radiation pressure and energy density become important greatly before a shock may be considered relativistic. For shocks propagating into normal air the shock velocity is one-tenth the speed of light at a Mach number of about 10^5, while radiation pressure is equal to material gas pressure behind the shock at a Match number of about 2×10^2. Also, radiative transport of energy may be important even though radiation pressure and energy density are negligible. For equilibrium between radiation and air at atmospheric density, radiation pressure is equal to material gas pressure at a temperature of about three million degrees absolute whereas radiative transport of enery becomes important at much lower temperatures. In order of significance (with increasing Mach number the effects are : radiative transfer, radiation energy density and pressure and finally relativistic nature of the shock.) In another work, we (1962) studied the decay of shock waves including the loss of energy by radiation for a perfect gas only. The ratio between the gas pressure and the radiation pressure is of the order of 4 : 1 in the relevant parts of the atmosphere of a B-2 star (Bhatnagar and Kushwaha, 1961 a). Therefore in the present note, the decay of the shock wave is being studied by taking into account all these radiation effects *i.e.*, radiation flux, radiation energy density and radiation pressure.

BASIC EQUATIONS

We have at the shock front :

(*a*) Equation of notion

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \cdot \frac{\partial p}{\partial x} = 0, \tag{1}$$

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(*b*) Equation of continuity

$$\frac{\rho}{\rho_0} \cdot \frac{\partial u}{\partial x} + \frac{1}{\rho c^2} \cdot \frac{\partial p}{\delta t} + \frac{\alpha u}{x} = 0, \tag{2}$$

where $u = \left(\frac{\partial \xi}{\partial t} \right)_x$ is the particle velocity, $p(= p_g + p_r)$ is the total pressure excees over the pressure $P_0(= P_{g_0} + P_{r_0})$ of the undisturbed fluid. The subscript g is used for the material gas and r for the radiation. ρ is the density at an instant and ρ_0 that of the undisturbed fluid. t is the time, and ξ the Eulerian coordinate at time t of an element of the fluid with Lagrangian coordinate x. c is the velocity of sound. The variation of P_0 with respect to x has been nenglected in comparison with that of p. The coefficient α takes the values 0, 1 and 2 for a plane, cylindrical and spherical shock wave respectively.

The Rankine—Hugoniot relations can be written as

$$p = \rho_0 \,.\, u/U, \tag{3}$$

$$\rho(U - u) = \rho_0 U, \tag{4}$$

$$\triangle \mathrm{H} = \frac{p}{2}\left(\frac{1}{\rho_0} + \frac{1}{\rho} \right). \tag{5}$$

Here U is the velocity of the shock front and $\triangle \mathrm{H}$ is the gain in specific enthalpy in crossing the shock front. Quantities ahead of the shock front are given by the subscript 0 and no subscript is used for the quantities across the front. On applying the hydrodynamical operator.

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\mathrm{U}} \frac{\partial}{\partial t} \tag{6}$$

to eqn. (3) one gets,

$$\frac{\partial u}{\partial t} + \mathrm{U} \frac{\partial u}{\partial x} - \frac{\mathrm{K}}{\rho_0} \cdot \frac{\partial p}{\partial x} - \frac{\mathrm{K}}{\rho_0 \mathrm{U}} \cdot \frac{\partial p}{\partial t} = 0, \tag{7}$$

where

$$k = \rho_0 \mathrm{U} \frac{du}{dp} = 1 - \frac{p}{\mathrm{U}} \cdot \frac{d\mathrm{U}}{dp}. \tag{8}$$

A fourth equation is necessary in addition to the three equations (1), (2) and (7) given above, to solve for the four partial deriatives of u and p. For this purpose consider the work done per unit area of the generating surface (including the loss of energy by radiation), which can be written as

$$u_0 a_0^{\alpha} = \int_{a_0}^{x} \rho_0 \xi^{\alpha} \left(\mathrm{E} + \frac{\mathrm{W}}{\rho_0} \right) d\,\xi_0 - \int_{a_0}^{x} \rho_0 \xi_0^{\alpha}\ \mathrm{F}\ d\,\xi_0 + \int_{t_p(x)}^{\infty} \xi^{\alpha}\ u'\ (\mathrm{P}_0 + p')\ dt, \tag{9}$$

where $\dot{E}$ and $\dot{W}$ are the specific internal energy increment of the fluid and the radiation energy density increment at P_0. u' and p' denote the particle valocity and excess pressure behind the shock front. $t_0(x)$ is the time of arrival of the shock at x, a_0 is the Lagrangian coordinate of the generating surface of the shockwave. F is the net energy loss by radiation per unit mass of the fluid. Introducing,

$$h = E + \frac{W}{\rho_0} + P_0 \triangle (1/\rho), \tag{10}$$

the specific enthalpy increment of an element of fluid on crossing the shock front and returning to pressure P_0 and then comparing it with the equation (9) for the limit $x \to \infty$, we get

$$D(x) = \int_{t_0(x)}^{\infty} \xi^{\alpha} u' p' \, dt \tag{10}$$

$$= \int_{x}^{\infty} \rho_0 \xi_0^{\alpha} h \, d\xi_0 - \int_{x}^{\infty} \rho_0 \xi_0^{\alpha} F \, d\xi_0 \tag{12}$$

In the Brinkley-Kirkwood theory of propagation of shockwaves, increase in enthalpy of the gas element is equal to the mechanical work done by the shock wave. However, in astrophysical cases where high temperatures are involved some of the energy is radiated away (which escape into intersteller space). Consequently the difference of enthalpy increment and this energy loss by radiation should be equated to the mechanical work done by the shock wave.

As in our previous work, from (11) (Bhatnagar and Kushwaha 1961 a) one can get

$$\frac{1}{u}\frac{\partial u}{\partial t} + \frac{1}{p}\frac{\partial p}{\partial t} + \frac{\alpha u}{x} = -\frac{x^{\alpha} p u v(x)}{D(x)}, \tag{13}$$

where $v(x)$ is a function depending on the form of the wave under consideration.

By eliminating $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ from eqs. (1), (2), (7) and (13) and solving for $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial t}$ we get the equation

$$\frac{dp}{dx} = -v \frac{x^{\alpha} p^3}{D(x)} \cdot \frac{l}{\rho_0 U^2} \cdot \frac{G}{2(k+1) - G} - \frac{\alpha p}{2x} \cdot \frac{4\left(\frac{\rho_0}{\rho}\right) + 2\left(1 - \frac{\rho_0}{\rho}\right)G}{2(k+1) - G}, \tag{14}$$

where,

$$G = 1 - \left(\frac{\rho_0 U}{\rho c}\right)^2. \tag{15}$$

From (12) one gets

$$\frac{dD}{dx} = -x^{\alpha}\rho_0 h + x^{\alpha}\rho_0 F. \tag{16}$$

In the present work, only the case of a plane shock wave, *i.e.* $\alpha = 0$ is considered. It has been shown earlier (Bhatnagar and Kushwaha, 1961 b) that the introduction of spherical curvature does not introduce any material difference in the computed velocities. If one makes $\alpha = 0$, then eqs. (14) and (16) reduce to

$$\frac{dp}{dx} = -\frac{\nu}{D(x)} \cdot \frac{p^3}{\rho_0 U^2} \cdot \frac{G}{2(k+1) - G}, \tag{17}$$

and

$$\frac{dD}{dx} = -\rho_0 h + \rho_0 F. \tag{18}$$

So far these equations are exact. For application to astrophysical problems these equations have to be numerically integrated by chosing appropriate initial values for the problem in hand.

APPROXIMATIONS

The Brinkley and Kirkwood (1947) theory of decay of a shock wave makes use of the fact that the irreversible processes all occur at the discontinuous shock front, so that these effects can be estimated purely from the strength of the shock, together with a simple assumption about the return of the material behind the front to the original equilibrium state. On the other hand, the energy carried by the disturbance can also be expressed in terms of the shock strength, together with an ass umption about the form of the wave, so in this way the dissipation can be related to the energy transporated. Thus a combination of thermodynamic reasoning with fairly plausible assumptions circumvents the necessity of explicitly working out the details of the dissipative processes in the shock transition zone. This method was applied to gas dynamic waves in the chromosphere by Schatman (1949) and Wevmann (1960) and to the β-cephei radial velocity variations by Odgers and Kushwaha, (1959) and later by Bhatnagar and Kushwaha (1961 *a* and *b*). The Rankine-Hugoniot relations are applied to connect the physical quantities across the shock front with those ahead of it. To obtain the physical quantities behind the shock from at an instant when the decay starts one has to make some assumption about the path of the gas element in the $P - \rho$ plane. In our earlier works (1961 *a* and *b*) we assumed that the mechanical decay started after the instaneous equalisation of temperature on the two sides of the shock front by large and fast radiation flux, but did not account for this flux in the relevant enery equation. Later (1962) an attempt was made by the same authors to include the loss of energy by radiation in our equation by neglecting the radiation pressure and energy density. It was done for the following two cases on the assumption that after crossing the shock front the radiative loss of energy takes place at constant volume till, (*i*) the gas element returns to the original adiabate, which consequently means a return to the original entropy (as the changes in ionisation behind the front were not considered) (*ii*) the temperature across the shock front becomes equal to that ahead of it.

The mechanical decay occuring after this was assumed to be due to adiabatic expansion to reach the original pressure. We have used the same assumptions in the present note also to estimate the loss of energy by radiation flux. We have taken the stellar matetial to be a mixture of a perfect gas and radiation in equilibrium. We can express the various physical quantities in terms of the shock strength, defined by the relation:

$$p = P_0 \, s. \tag{19}$$

The relation between the densities on the two sides of the shock front can be found as follows. If the expression for enthalpy increment for a mixture of perfect gas and radiation,

$$\Delta H = \frac{\gamma}{\gamma - 1}\left(\frac{Pg}{\rho} - \frac{Pg_0}{\rho_0}\right) + 4\left(\frac{Pr}{\rho} - \frac{Pr_0}{\rho_0}\right) \tag{20}$$

is equated with the corresponding expression in eqn. (5) then one gets

$$Z \equiv \frac{\rho_0}{\rho} = \frac{M + s}{(N\beta + 8)(1 + s) - s}, \tag{21}$$

where,

$$M = 2\left[\left(\frac{\gamma}{\gamma - 1} - 4\right)\beta_0 + 4\right], \tag{22}$$

$$N = 2\left(\frac{\gamma}{\gamma - 1} - 4\right), \tag{23}$$

$$\beta = Pg/P, \quad \beta_0 = Pg_0/P_0, \tag{24}$$

$$P_g = P_{g_0} + p_{g_0}; \quad Pr = P_{r_0} + p_r. \tag{25}$$

From eqs (24) and (25) one can easily get

$$\frac{1 - \beta}{\beta^4} = \frac{1 - \beta_0}{\beta_0^4}(1 + s)^3 . Z^4, \tag{26}$$

which gives the dependence of β on s and Z as the shock decays.

Using the Rankine-Hugoniot relations (3) and (4) one can write u, U, G, k, c and c_0 as follows :

$$u = \frac{C_0}{\Gamma^{1/2}} s^{1/2}(1 - Z)^{1/2}, \tag{27}$$

$$U = \frac{C_0}{\Gamma_0^{1/2}} s^{1/2}(1 - Z)^{-1/2}, \tag{28}$$

$$G = 1 - \frac{Z}{1 - Z} \cdot \frac{s}{\Gamma(1 + s)}, \tag{29}$$

$$k = \tfrac{1}{2}\left[1 - \frac{s}{1 - Z} \cdot \frac{dZ}{ds} \right], \tag{30}$$

$$c_0^2 = \Gamma_0 \cdot \frac{P_0}{\rho_0}, \tag{31}$$

$$c^2 = \frac{\Gamma}{\Gamma_0} \cdot c_0^2 \cdot Z\,(1 + s), \tag{32}$$

$$\Gamma = \beta + \frac{(4 - 3\,\beta)^2\,(\gamma - 1)}{\beta + 12\,(\gamma - 1)\,(1 - \beta)}, \tag{33}$$

$$\Gamma_0 = \beta_0 + \frac{(4 - 3\beta_0)^2\,(\gamma - 1)}{\beta_0 + 12\,(\gamma - 1)\,(1 - \beta_0)}, \tag{34}$$

where,

$$\frac{dZ}{ds} = \frac{(N\,\beta + 7)\,(1 - M) + 1}{[(N\,\beta + 8)\,(1 + s) - s]^2} \tag{35}$$

$$- \frac{N\,\beta\,(1 - \beta)}{[N\,(\beta + 8)\,(1 + s) - s]^2} \cdot \frac{(1 + s)\,[(N\,\beta + 7)\,(3\,s + 4 - M) + 4] + 3\,(M + s)}{(1 + s)\,(24\,\beta - N\,\beta^2 - 32) + (4 - 3\,\beta)\,s}.$$

We can determine the energy loss by radiation for following two cases.

Case A :

On crossing the shock front the steller material radiates at constant volume till the entropy on the two sides has the same value *i.e.* it returns to the original adiabate specified by Γ_0 and thereafter it expands adiabatically to the original pressure.

Case B :

The radiative loss of energy behind the shock front takes place at constant volume till the temperature on the two sides of the front is equalised and thereafter it expands adiabatically to the original pressure. We first evaluate in detail h and F for the case A and then give the corresponding expressions for the case B.

CASE A

On crossing the shock front the internal energy of the gas and the radiation energy density increase. Then the material radiates some energy F per unit mass and finally does adiabatic work in expanding back to the original pressure. Therefore,

$$F = \Delta\, e - \Delta\, w \tag{36}$$

where $\Delta\, e$ is the gain in specific internal energy and the radiation energy per unit mass on crossing the shock front. $\Delta\, w$ is the work done per unit mass of the stellar

matter in adiabatic expansion to the original equilibrium pressure. The terms on the right side of eq. (36) can be expressed.

$$\Delta e = \frac{P_o}{\rho_o}(1+s)\cdot \zeta \cdot \left[\frac{\beta}{\gamma-1}+3(1-\beta)\right]-\frac{P_o}{\rho_o}\left[\frac{\beta_o}{\gamma-1}+3(1-\beta_o)\right], \quad (37)$$

and

$$\Delta w = \frac{P_o}{\rho_o}\cdot\frac{1}{\zeta^{\Gamma_o-1}}\left[\frac{\beta_o}{\gamma-1}+3(1-\beta_o)\right]-\frac{P_o}{\rho_o}\left[\frac{\beta_o}{\gamma-1}+3(1-\beta_o)\right] \quad (38)$$

Therefore,

$$F=\frac{P_o}{\rho_o}(1+s)\zeta\cdot\left[\frac{\beta}{\gamma-1}+3(1-\beta)\right]-\frac{P_o}{\rho_o}\cdot\zeta^{1-\Gamma_o}\left[\frac{\beta_o}{\gamma-1}+3(1-\beta_o)\right]. \quad (39)$$

The enthalpy increment can be written as

$$\begin{aligned} h &= \Delta H + \Delta H^* \\ &= \frac{P_o}{\rho_o}\cdot\frac{s}{2}\cdot(1+\zeta)-\frac{P_o}{\rho_o}\cdot\zeta^{1-\Gamma_o}\left[\frac{\gamma}{\gamma-1}\beta_o+4(1-\beta_o)\right] \\ &\quad + \frac{P_o}{\rho_o}\left[\frac{\gamma}{\gamma-1}\beta_o+4(1-\beta_o)\right], \end{aligned} \quad (40)$$

where, ΔH^* is the change in enthalpy during the adiabatic expansion to the final pressure P_o.

Using eqs. (27) to (40) in eqs. (17) and (18) we get

$$\frac{ds}{dx} = -\frac{\nu}{D(x)}P_o\cdot s^2\cdot(1-\zeta)\cdot\frac{\Gamma(1+s)(1-\zeta)-\zeta s}{\Gamma(1+s)\left[2(1-\zeta)-s\cdot\frac{d\zeta}{ds}\right]+\zeta s}, \quad (41)$$

and

$$\frac{dD}{dx} = P_o\cdot\left[-\frac{s}{2}(1+\zeta)-\frac{M}{2}+\zeta(1+s)\left\{\frac{4-3\gamma}{\gamma-1}\beta+3\right\}+\zeta^{1-\Gamma_o}\right]. \quad (42)$$

It is convenient to introduce in eqs. (41) and (42) two dimensionless quantities ζ and y defined by the relations

$$x = R\cdot y, \quad (43)$$

and

$$D(x) = P_o\cdot R\cdot \zeta, \quad (44)$$

where R is the stellars radius, In terms of these, equations (41) and (42) reduce to

$$\frac{ds}{dy} = -\frac{v}{\zeta} \cdot s^2 \cdot (1 - Z) \cdot \frac{\Gamma(1+s)(1-Z) - Z s}{\Gamma(1+s)\left[2(1-Z) - s \cdot \frac{dZ}{ds}\right] + Z s}, \tag{45}$$

and

$$\frac{d\zeta}{dy} = -\frac{s}{2} \cdot (1 + Z) - \frac{M}{2} + Z(1+s)\left[\frac{4-3\gamma}{\gamma - 1}\beta + 3\right] + Z^{1-\Gamma_o}. \tag{46}$$

CASE B

Following the similar procedure we find that in this case, $\triangle e$ remains the same, but $\triangle w$ becomes

$$\triangle w = \frac{P_o}{\rho_o} \cdot \frac{\beta_o}{\beta_1} \cdot \left[\frac{\beta_1}{\gamma - 1} + 3(1-\beta_1)\right] \cdot \left[1 - \left(\frac{\beta_1}{\beta_o} \times Z\right)^{1 - 1/\Gamma_1}\right], \tag{47}$$

where

$$\beta_1 = \frac{\beta_o}{\beta_o(1 - Z) + Z}, \tag{48}$$

and

$$\Gamma_1 = \beta_1 + \frac{(4 - 3\beta_1)^2(\gamma - 1)}{\beta_1 + 12(\gamma - 1)(1 - \beta_1)}. \tag{49}$$

Therefore,

$$F = \frac{P_o}{\rho_o}(1+s) \cdot Z\left[\frac{\beta}{\gamma - 1} + 3(1 - \beta)\right] - \frac{P_o}{\rho_o}\left[\frac{\beta_o}{\gamma - 1} + 3(1 - \beta_o)\right]$$

$$- \frac{P_o}{\rho_o}\gamma\frac{\beta_o}{\beta_1}\left[\frac{\beta_1}{\gamma - 1} + 3(1 - \beta_1)\right] \cdot \left[1 - \left(\frac{\beta_1}{\beta_o} \cdot Z\right)^{1 - 1/\Gamma_1}\right], \tag{50}$$

and

$$h = \frac{P_o}{\rho_o} \cdot \frac{s}{2}(1 + Z) - \frac{P_o}{\rho_o} \cdot \frac{\beta_o}{\beta_1}\left[\frac{\gamma}{\gamma - 1}\beta_1 + 4(1 - \beta_1)\right].$$

$$\left[1 - \left(\frac{\beta_1}{\beta_o} \cdot Z\right)^{1 - 1/\Gamma_1}\right]. \tag{51}$$

The eqn. (41) and consequently (45) remain the same but eqn. (42) becomes

$$\frac{dD}{dx} = P_o\Big[- \frac{s}{2}(1+\mathcal{Z}) + \frac{\beta_0}{\beta_1}\Big\{1 - \Big(\frac{\beta_1}{\beta_0}\cdot\mathcal{Z}\Big)^{1-1/\Gamma_1}\Big\}$$

$$+ \mathcal{Z}(1+s)\Big\{\frac{\beta}{\gamma-1} + 3(1-\beta)\Big\} - \Big\{\frac{\beta_0}{\gamma-1} + 3(1-\beta_0)\Big\}\Big], \tag{52}$$

which can be written in terms of ζ and y as

$$\frac{d\zeta}{dy} = -\frac{s}{2}(1+\mathcal{Z}) + \frac{\beta_0}{\beta_1}\Big[1 - \Big(\frac{\beta_1}{\beta_0}\mathcal{Z}\Big)^{1-1/\Gamma_1}\Big]$$

$$+ \mathcal{Z}(1+s)\Big[\frac{\beta}{\gamma-1} + 3(1-\beta)\Big] - \Big[\frac{\beta_0}{\gamma-1} + 3(1-\beta_0)\Big] \tag{53}$$

INITIAL VALUES AND NUMERICAL INTEGRATIONS

For application to actual problems one has to integrate numeracally the equations (45), (46) for the case A or (45) and (53) for the case B. In the present note we compute theoretically the radial velocity variations of β -cephei star BW Vul. For this star we take $\gamma = 5/3$. The initial values $\beta_0 = 0{\cdot}8$, $\Gamma_0 = 1{\cdot}5$ and $\zeta = 1$ are as before (Bhatnagar and Kushwaha, 1961 *a*). ν is a function depending on the form of the wave under consideration. Its initial value varies between 2/3 and 1. For asymptotic behaviour its value was given by Brinkley and Kirkwood to be 2/3, which has been adapted here. Using the initial value of particle velocity - 130 Km/sec. (Odgers and Kushwaha, 1959), we solve equations (21), (26) and (27) simultaneously to determine the initial values of s, $\mathcal{Z}$ and β. These give $s \simeq 48{\cdot}3$; $\rho/\rho_0 = 5{\cdot}5340$ and $\beta = 0{\cdot}3228$. The equations (45), (46) for case A or (45) and (53) for the case B, have been numerically integrated following Härm and Schwarzschild (1955) scheme At every step β and $\mathcal{Z}$ are determined from eqs. (21) and (26) for the value of s.

DISCUSSION AND CONCLUSION

The value of ζ, s, u/C_0 and U/C_0 obtained from the integrations as the shock propagates are given in tables 1 and 2 for the two cases under consideration.

TABLE 1

Decay of a shock wave for case A

y	ζ	s	u/C_0	U/C_0
0·00	1·0000	48·3000	5·11	6·24
0·01	0·9384	43·7086	4·85	5·94
0·02	0·8830	39·7272	4·62	5·68
0·03	0·8331	36·2546	4·40	5·44
0·04	0·7880	33·2128	4·21	5·21
0·05	0·7471	30·5338	4·02	5·01
0·06	0·7097	28·1654	3·86	4·82
0·07	0·6756	26·0613	3·70	4·65
0·08	0·6444	24·1867	3·56	4·49
0·09	0·6158	22·5096	3·42	4·34
0·10	0·5895	21·0036	3·30	4·20
0·11	0·5653	19·6472	3·18	4·07
0·12	0·5429	18·4222	3·08	3·95
0·13	0·5222	17·3121	2·97	3·84
0·14	0·5030	16·3037	2·88	3·74
0·15	0·4852	15·3850	2·79	3·64
0·16	0·4686	14·5460	2·71	3·55
0·18	0·4388	13·0744	2·55	3·38
0·20	0·4127	11·8306	2·41	3·23

TABLE 2

Decay of a shock wave for case B

y	ζ	s	u/C_0	U/C_0
0·00	1·0000	48·3000	5·11	6·24
0·01	0·9282	43·6858	4·85	5·94
0·02	0·8629	39·6461	4·61	5·67
0·03	0·8033	36·0941	4·39	5·43
0·04	0·7488	32·9575	4·19	5·19
0·05	0·6988	30·1732	4·00	4·98
0·06	0·6528	27·6977	3·82	4·78
0·07	0·6104	25·4851	3·66	4·60
0·08	0·5712	23·5018	3·50	4·43
0·09	0·5350	21·7188	3·35	4·27
0·10	0·5013	20·1107	3·22	4·12
0·11	0·4701	18·6567	3·10	3·98
0·12	0·4410	17·3384	2·98	3·84
0·13	0·4139	16·1402	2·86	3·72
0·14	0·3887	15·0486	2·76	3·60
0·16	0·3429	13·1396	2·56	3·39
0·18	0·3027	11·5340	2·38	3·20
0·20	0·2673	10·1738	2·22	3·02

If one plots U/C_0 against y, the relation is found to be almost linear for $0 \leqslant y \leqslant 0·15$ and we can relate U/C_0 and y by a linear equation, which on integration gives time. The values of computed velocities as a function of time are given in Table 3.

TABLE 3

Case A		Case B	
t (days)	u (km/sec)	t (days)	u (km/sec)
0	111·82	0	111·52
0·004270	106·81	0·005313	106·38
0·008654	102·20		
0·013159	97·94	0·010863	101·56
0·017795	94·01	0·016671	97·08
0·022567	90·36		
0·027483	86·97	0·022762	92·90
0·032553	83·82	0·029165	88·99
0·037787	80·88	0·035913	85·19
0·043196	78·13	0·043047	81·89
0·048791	75·55		
0·054587	73·14	0·050613	78·66
0·060597	70·88	0·058666	75·62

In fig. 1 we have plotted the observed radial velocity from Odgors (1955) ob-

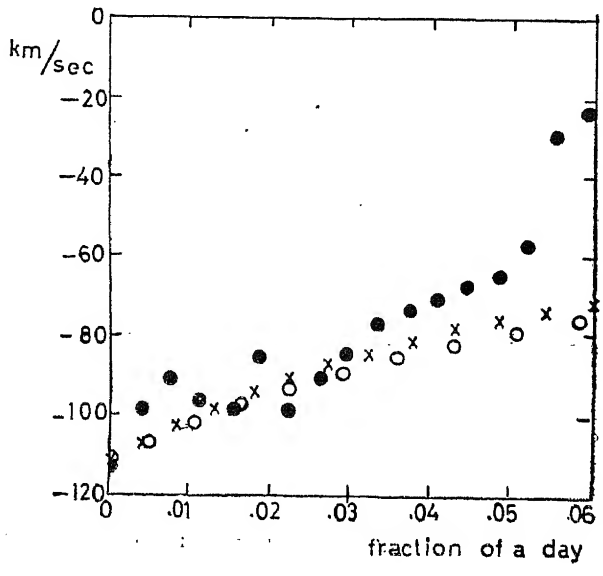

Fig. 1. Comparison of the observed and computed radial velocity curves of BW Vul. The computed velocities are compared with those observed during the cycle beginning J. D. 2, 435.009, 7,600. The observed velocities have been corrected for fore-shortening. X give the computed velocities for case A and O give those for the case B, whereas ● show the observed velocities.

servations. It is seen that for this particular cycle the initial plotted value of velocity after applying corrections for fore-shortening and limb-darkening is of the order of $-$ 112 km/sec. Hence we have shifted the time scale of our computed velocity curve in such a manner that time scale starts at a velocity $\approx -$ 112 km/sec. We find that the computed radial velocity curves are some what lower than the observed one and time of decay is a little larger than the observed value. For this a qualitative explanation can be given as follows. We have supposed the shock front to be transparent to the radiation below so that the observations refer to some sort of integrated motion of the matter behind the front. After an element of gas has received its initial outward velocity it then moves under effective gravity (not negligible in these stars but not included in our considerations). In the lower layers of the atmosphere where the shock wave is strong the instantaneous velocity imparted by the shock dominates the observed motions, since the even lower layers are largely obscurred by the much increased opacity. However, at higher levels where the energy of the wave has largely been dissipated the observed motion consists mostly of material under the effective gravity forces only. On this view as the shock progresses, it should rapidly decay due to gravitational forces. The computed velocities which are having large negative value will be reduced and may give the required agreement. We have neglected the variations of pressure and density with space in the undisturbed atmosphere. As the total distance traversed by the shock wave is large, this inhomogenity may also have some effect on the computed rate of decay, The study of the effect of inhomogenity is in hand. Finally we have estimated the loss of energy by radiation just intuitively without giving any specific mechanism for it. In the previous work (1962) the initial jumps in temperature and density across the front were $T/T_0 \approx 1{\cdot}6$ and $\rho/\rho_0 \approx 3{\cdot}8$. In the present case they are changed to $T/T_0 \approx 3{\cdot}6$ and $\rho/\rho_0 \approx 5{\cdot}5$. This change is due to the effect of radiation pressure and density being taken into account in the present work. These changes in jumps are in closer agreement with the ideas of Abt (1955) and Save doff (1954). In figs. (2), (3) and (4) we have plotted P/P_0,

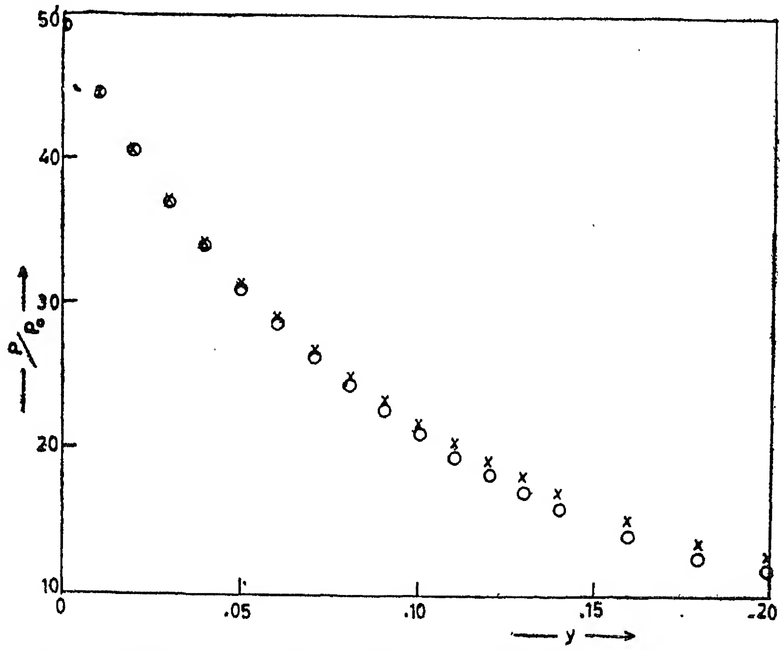

Fig. 2. Pressure versus distance curve across the shock front. X give the case A and ⊙ give Case B.

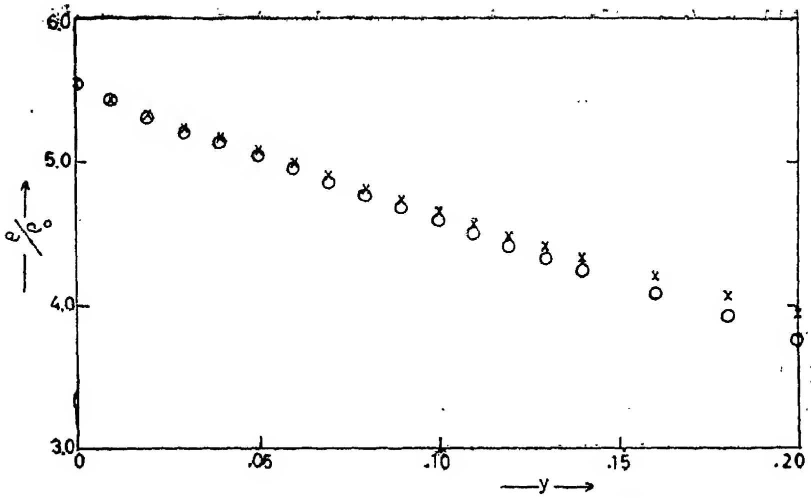

Fig. 3. Densities across the shock front have been plotted against distance. X give case A and O give case B.

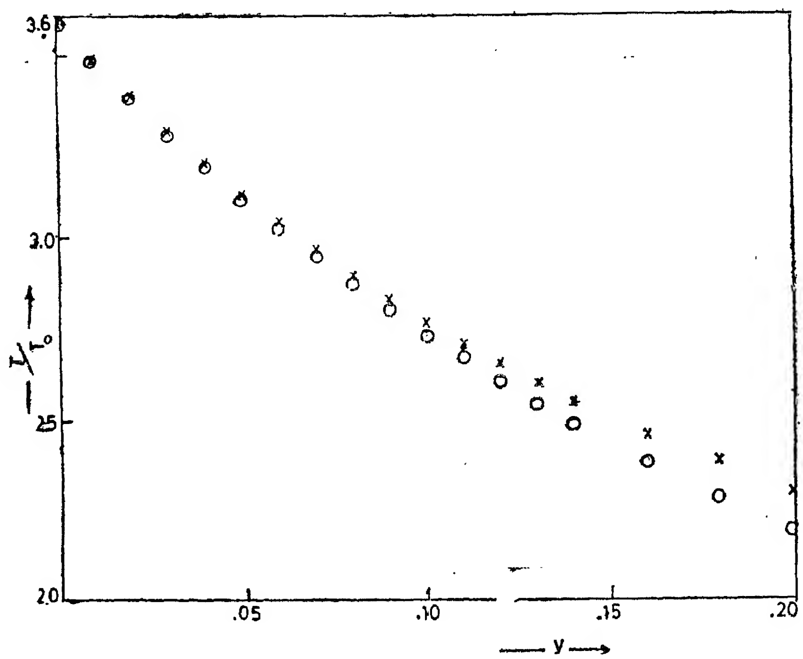

Fig. 4. Temperature across the shock front versus distance curve. X give case A and O give case B.

ρ/ρ_0 and T/T_0 against y (which have been tabulated in Tables 4) to study roughly the variations of pressure, temperature and density across the shock front with space.

TABLE 4

	Case B			Case A		
y	ρ/ρ_0	T/T_0	P/P_0	ρ/ρ_0	T/T_0	P/P_0
0·00	5·5340	3·5946	49·30	5·5340	3·5946	49·30
0·01	5·4434	3·4846	44·69	5·4437	3·4854	44·71
0·02	5·3505	3·3824	40·65	5·3533	3·3836	40·73
0·03	5·2576	3·2842	37·09	5·2632	3·2888	37·25
0·04	5·1645	3·1918	33·96	5·1733	3·2000	34·21
0·05	5·0713	3·1052	31·17	5·0839	3·1161	31·53
0·06	4·9788	3·0225	28·70	4·9975	3·0369	29·17
0·07	4·8862	2·9432	26·49	4·9116	2·9629	27·06
0·08	4·7943	2·8673	24·50	4·8263	2·8938	25·19
0·10	4·7030	2·7948	22·72	4·7438	2·8273	23·51
0·10	4·6126	2·7266	21·11	4·6642	2·7645	22·00
0·11	4·5230	2·6609	19·66	4·5851	2·7059	20·65
0·12	4·4320	2·5985	18·34	4·5086	2·6499	19·42
0·13	4·3465	2·5394	17·14	4·4326	2·5975	18·31
0·14	4·2600	2·4818	16·05	4·3592	2·5474	17·30
0·16	4·0910	2·3745	14·14	4·2176	2·4549	15·55
0·18	3·9263	2·2778	12·53	4·0850	2·3713	14·07
0·20	3·7680	2·1875	11·17	3·9588	2·2963	12·83

It may be noted that there is not much difference in the computed values for cases A and B. Density and temperature gradients are almost linear.

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SOME PROBLEMS ON GENERAL UNIQUENESS AND SUCCESSIVE APPROXIMATIONS

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ABSTRACT

The problems on General Uniqueness and successive approximations was considered by F. Brauer (1959) in a sufficiently general way so as to include the results of F. Brauer and S. Sternberg (1958) which are generalisations of Coddington and Levinson and E. A. Kamke. In this paper we shall prove a pair of theorems under conditions which are generalisations again.

§1. We know that the uniqueness of solutions and the convergence of successive approximations are logically independent in the theory of Differential equations cf. [2] and [6]. In this paper, we shall prove a pair of theorems generalising the results of [2] and [5] which are generalisation of Coddington and Levinson [3] and Kamke [4]. The generalisations here are in a different approach as pointed out in [1].

§2. Consider the differential system

$$(1) \qquad x'(t) = f(x(t), t) \qquad x(o) = o \qquad \left(' = \frac{d}{dt} \right)$$

Where x and f are n-dimensional vectors.

Let $\Phi(x(t), t)$ be a function, defined for vectors x and real t with non-negative real values, which is continuous in (x, t) has one sided partial derivatives with respect to t and the components of x such that $\Phi(x, t) = o$ implies $x = o$. Let ϕ_t ϕ_x be the partial derivatives of Φ with respect to t and x respectively.

Let $\omega_i(r, t)$ $(i = 1, 2)$ be continuous, non-negative functions defined for $o < t < a$, $r \geqslant o$ which are monotone non-decreasing in r for each fixed t. We will always assume that f satisfies the pair of conditions :

$$\phi_t \left\{ (x_1 - x_2)^{\alpha}, t \right\} + \alpha (x_1 - x_2)^{\alpha - 1} \phi_x \left\{ (x_1 - x_2)^{\alpha}, t \right\}$$

$$(2) \qquad \left[f(x_1, t) - f(x_2, t) \right] \leqslant \omega_i \left[\phi \left\{ (x_1 - x_2)^{\alpha}, t \right\}, t \right]_{i = 1, 2}$$

where $\alpha \geqslant 1$ for $(t_1 x_1)$ and $(t_1 x_2)$ in the region $o < t < a$, $| x | < b$

Theorem:1 :—Let $f(x,t)$ be continuous and satisfy (2) in the region $o < t < a$, $| x | \leqslant b$. Suppose A(t) and B(t) are functions on $o \leqslant t < a$ with A(o) = B(o) = O, such that

$$(3) \qquad \lim_{t \to o} \frac{A(t)}{B(t)} = O$$

Suppose also that all solutions $u(t)$ of

(4) $$u'(t) = \omega_1(u(t), t)$$

with $u(O) = O$ obey $u(t) \leqslant A(t)$ on $o \leqslant t < a$, and that the only solution $v(t)$ of

(5) $$v'(t) = \omega_2(v(t), t)$$

on $o \leqslant t < a$ such that

$$\lim_{t \leftarrow o} \frac{v(t)}{B(t)} = O$$

is the trivial solution. Then there is at most one solution of (1) on $o \leqslant t < a$.

Proof :—Suppose $x_1(t)$ and $x_2(t)$ be two solutions of (1) on $o \leqslant t < a$, if possible, let

$$\mu(t) = x_1(t) - x_2(t), \text{ so that}$$

(6) $$\mu'(t) = x_1'(t) - x_2'(t)$$

Let $m(t) = \phi\left(\mu^{\alpha}(t), t\right)$, $m^*(t) = \lim_{h \to o} \text{Sup} \left[m(t) - m(t-h) \right]/h$

then

$$\frac{m(t) - m(t-h)}{h} = \left[\Phi\left(\mu^{\alpha}(t), t\right) - \Phi\left(\mu^{\alpha}(t-h), t\right) + \Phi\left(\mu^{\alpha}(t-h), t\right) - \Phi\left(\mu^{\alpha}(t-h), t-h\right) \right]/h$$

Since Φ has one sided partial derivatives, the left hand side of the above equation is bounded by a sum of the form

$$\left\{ \alpha\, \mu^{\alpha-1}(t) \cdot \phi_x\left(\mu^{\alpha}(t), t\right) + \varepsilon_1 \right\} \left[\mu(t) - \mu(t-h) \right]/h + \Phi_t\left(\mu^{\alpha}(t), t\right) + \varepsilon_2$$

where ε's $\to o$ as $h \to v$

Now allowing $h \to O$, we obtain in view of (2) and (6) that

(7) $$m^*(t) \leqslant \omega_1(m(t), t)$$

Let us assume that there exists a ξ, $o < \xi < a$ such that

(8) $$m(\xi) > A(\xi)$$

Then there is a solution $u(t)$ of (4) passing through the point $(\xi, m(\xi))$ and existing on some interval to the left of ξ. As far to the left of ξ as $u(t)$ exists it satisfies

(9) $$u(t) \leqslant m(t)$$

To prove (9), consider the solution $u(t, \varepsilon)$ of the equation

(10) $$u' = \omega_1(u, t) + \varepsilon \qquad u(\xi) = m(\xi)$$

which we know exists for sufficiently small $\varepsilon > o$, as far to the left of ξ as $u(t)$ exists. Moreover

$$\lim_{\varepsilon \to o} u(t, \varepsilon) = u(t) \quad [4]$$

Thus to prove (9), it suffies to establish that

(11) $$u(t, \varepsilon) \leqslant m(t)$$

for all $\varepsilon > o$ and all solutions of (10). If this were false, there must be a least upper bound η of numbers $t \leqslant \xi$ for which (11) is not true. Since $m(\xi) = u(\xi) = u(\xi, \varepsilon)$ and the functions $m(t)$, $u(t, \varepsilon)$ are continuous, it follows that

(12) $$m(\eta) = u(\eta, \varepsilon) \qquad m^*(\eta) \geqslant u'(\eta, \varepsilon)$$

Then

$$\begin{aligned}\omega_1(\eta, m(\eta)) + \varepsilon &= \omega_1(\eta, u(\eta, \varepsilon)) + \varepsilon \\ &= u'(\eta, \varepsilon) \\ &\leqslant m^*(\eta) \\ &\leqslant \omega_1(\eta, m(\eta))\end{aligned}$$

using (7), (10), (12). This contradiction proves (11), which implies (9).

The solution $u(t)$ can be continued to $t = O$. If $u(c) = o$ for some c, $o < c < \xi$, we can effect the continuation by defining $u(t) = o$ for $o < t < c$; otherwise (9) ensures the possibility of continuation. Since $m(O) = O$, $\lim_{t \to o} u(t) = c$ and we define $u(o) = O$. Now we have a solution $u(t)$ of (4) with $u(o) = O$ and by hypothesis $u(t) \leqslant A(t)$. However, in view of (8) and the definition of $u(t)$, we have $u(\xi) > A(\xi)$, a contradiction which proves

(13) $$m(t) \leqslant A(t) \qquad o \leqslant t < a$$

To prove the uniqueness of solutions, we must show that $m(t)$ vanishes identically on $o \leqslant t < a$. To complete the proof, we must show that this implied by (13). Proceeding as before but using ω_2 in the place of ω_1, we obtain

$$m^*(t) \leqslant \omega_2(m(t), t)$$

The assumption that $m(\Gamma) > o$ for some Γ, $o < \Gamma < a$ yields by the same argument as before, a solution $v(t)$ of (5) on $o \leqslant t \leqslant \Gamma$ such that $v(\Gamma) = m(\Gamma)$, $o \leqslant v(t) \leqslant m(t)$, $v(o) = o$

Then

$$O \leqslant \lim_{t \to o} \frac{v(t)}{B(t)} \leqslant \lim_{t \to o} \frac{m(t)}{B(t)} \leqslant \lim_{t \to o} \frac{A(t)}{B(t)} = O$$

using (3) and (13). But by hypothesis, this implies that $v(t)$ is identically zero, which contradiction $v(\Gamma) = m(\Gamma) > o$ and therefore $m(t)$ vanishes identically on $o \leqslant t < a$ which completes the proof of the theorem since $\phi(x, t) = o$ implies $x = o$
Suppose, if we take

$$\omega_1 (r, t) = \omega (r, t)$$

$$\omega_2 (r, t) = 2 M$$

where M is a bound for $| f(x, t) |$ in $o < t < a, | x | \leqslant b$

Further, if we put $B(t) = t$ and let $A(t)$ be a solution of (4) with $A(o) = A'(o) = o$, we get the results of [5].

The results due to F. Brauer and S. Sternberg referred to earlier is for a system of differential equations but with $\alpha = 1$.

Again putting $\Phi (x, t) = | x |$, we obtain the results of general uniqueness theorem of Kamke [4].

§3. The successive approximations to the solution of (1) are defined by

$$x_{j+1} (t) = \int_0^t f(x_j (S), S) \, d S \qquad x_0 (t) = o \tag{14}$$

$$(j = o, 1, \ldots)$$

In the present section, we shall impose two additional conditions on Φ to guarantee the convergence of successive approximations. We assume that Φ has properties required in § 2, and also obeys

$$\Phi\left(\int_{t-h}^{t} \alpha \mu_{j+1}^{\alpha-1} (S) \left[f(x_{j+1}(S), S) - f(x_j(S), S) \right] d S, t \right)$$

$$\leqslant \int_{t-h}^{t} \Phi_t (\mu_j^{\alpha} (S), S) \, d S + \int_{t-h}^{t} \alpha \mu_j^{\alpha-1} (S) \, \Phi_x (\mu_j^{\alpha} (S), S) \left[f(x_{j+1} (S), S) - f(x_j (S),(S)) \right] d S \tag{15a}$$

and

$$\Phi (max (t_1, t_2), x_{j+1} + x_j \leqslant \Phi (x_{j+1}, t_1) + \Phi (x_j, t_2) \tag{15b}$$

Where $\alpha \geqslant 1$ for any continuous functions of f and $\mu_j = x_{j+1} - x_j$

Theorem : 2 : Let Φ be as in § 2, with the additional properties (15a) and (15b). Let $f(x, t)$ be continuous in the region $o < t < a$, $| x | \leqslant t$ and bounded in

norm by M in this region. Suppose the hypothesis of theorem 1 are satisfied, then the successive approximations (14) converage uniformly on the interval $O \leqslant t < \beta$. where $\beta = \min(a, b/M)$ to the unique solution of (1).

Proof:—It is easy to verify that since f is continuous and hence bounded, the sequence $x_j\ (t)$ of successive approximations is uniformly bounded and equi-continuous in the Euclidean norm on some interval. It follows that there is a subsequence $x_{j\,k}(t)$ which converges uniformly on this interval to the solution $x\,(t)$. Since

$$x_{j\,k+1}\ (t) = \int_0^t f(x_{j\,k}(S), S)\, d\,S$$

$x_{j\,k+1}\ (t)$ converges uniformly to a solution $x^*(t)$. We shall prove

$$\Phi\,(x_{j+1}(t) - x_j\ (t), t) \to O \tag{16}$$

on this interval which imply $x(t) = x^*(t)$ so that $x(t)$ is a solution of (1). Since the solution is unique, every convergent subsequence converges to $x(t)$ and it follows that the original sequence converges to $x(t)$. Since this sequence is uniformly bounded and equi-continuous, the convergence is uniform in the Euclidean norm which implies by the continuity of Φ, that $\Phi\,(x(t) - x_j\ (t), t)$ converges uniformly to zero on some interval.

To prove (16), since

$$\mu_j\ (t) = x_{j+1}(t) - x_j\ (t), \text{ therefore}$$

$$\mu'_j\ (t) = x'_{j+1} - x'_j$$

and let $m(t) = \lim \operatorname{Sup} \Phi\,(\mu_j^\alpha\ (t), t)$ as $j \to \infty$

then $m(o) = O$ and $m(t)$ is continuous, since it is the upper limit of a uniformly bounded equi-continuous sequence of functions. We have

$$\mu_{j+1}^\alpha\ (t) - \mu_{j+1}^\alpha\ (t - h) = \int_{t-h}^t d\,(\mu_{j+1}^\alpha(S))$$

$$= \int_{t-h}^{t} \alpha \mu_{j+1}^{\alpha-1}(S) \left[f(x_{j+1}(S), S) - f(x_j(S), S) \right] dS$$

$$\therefore \Phi(\mu_{j+1}^{\alpha}(t) - \mu_{j+1}^{\alpha}(t-h), t) = \Phi\Big(\int_{t-h}^{t} \alpha \mu_{j+1}^{\alpha-1}(S) \left[f x_{j+1}(S), S) \right.$$

$$\left. - f(x_j(S), (S) \right] dS, t \Big)$$

$$\leqslant \int_{t-h}^{t} \Phi_t (\mu_j^{\alpha}(S), S)\, dS + \int_{t-h}^{t} \alpha \mu_j^{\alpha-1}(S)\, \Phi_x (\mu_j^{\alpha}(S), S) \left[f(x_{j+1}(S), S) \right.$$

$$\left. - f(x_j(S), S) \right] dS$$

(17) $$\leqslant \int_{t-h}^{t} \omega_1 \left[\Phi(\mu_j^{\alpha}(S), S), S \right] dS \qquad \text{from (15a) and (2)}$$

given any $\delta > O$, there exists an integer $N(\delta)$ independent of s and j such that

(18) $$\Phi(\mu_j^{\alpha}(S), S) < m(S) + \delta \qquad j \geqslant N(\delta)$$

This follows from the fact that m is uniformly continuous and μ_j^{α} is equi-continuous. Since $\omega_1(r, t)$ is assumed monotone non-decreasing in r, it follows from (18) that

$$\int_{t-h}^{t} \omega_1 \left[\Phi(\mu_j^{\alpha}(S), S), S \right] dS \leqslant \int_{t-h}^{t} \omega_1 \left[m(S) + \delta, S \right] dS$$

using (17), we obtain

(19) $$\Phi(\mu_{j+1}^{\alpha}(t) - \mu_{j+1}^{\alpha}(t-h), t) \leqslant \int_{t-h}^{t} \omega_1 \left[m(S) + \delta, S \right] dS_j > N(\delta)$$

From the definition of $m(t)$ and (19) and (15b), it is easy to verify that..

(20) $$m(t) - m(t-h) \leqslant \int_{t-h}^{t} \omega_1 \left[m(S) + \delta, S \right] dS$$

Since $\omega_1(r, t)$ is continuous in r

$$\omega_1 \left[m(S) + \delta, S \right] \to \omega_1 \left[m(S), S \right] \text{ as } \delta = O$$

and this together with (20) yields

(21) $$m(t) - m(t-h) \leqslant \int_{t-h}^{t} \omega_1 (m(S), S)\, dS$$

This follows that

$$m^*(t) = \limsup_{h \to o} \left[m(t) - m(t-h) \right] /h$$

$$\leqslant \omega_1 \left[m(t), t \right]$$

The argument used in the proof of theorem 1 starting with (7) proves

(22) $$m(t) \leqslant A(t)$$

To complete the proof, we must show that (22) implies $m(t) \equiv O$. The argument is much the same as the last stage of the proof of theorem 1. Suppose $m(\Gamma) > o$ for some Γ, $O < \Gamma < \beta$. A repetition of the first part of the proof using ω_2 in the place of ω_1, gives a solution $v(t)$ of (5) on $O \leqslant t \leqslant \Gamma$ such that $v(\Gamma) = m(\Gamma)$, $O \leqslant v(t) \leqslant m(t)$, $v(o) = O$

Then $$O \leqslant \lim_{t \to o} \frac{v(t)}{B(t)} \leqslant \lim_{t \to o} \frac{m(t)}{B(t)} \leqslant \lim_{t \to o} \frac{A(t)}{B(t)} = O$$

using (3) and (22). By hypothesis this implies that $v(t)$ vanishes identically, contradicting $v(\Gamma) = m(\Gamma) > o$ and it implies $m(t) = O$, $o \leqslant t < \beta$ which proves the theorem,

Suppose if we take

$$\omega_1 (r, t) = \omega (r, t)$$
$$\omega_2 (r, t) = 2\, M$$

where M is a bound for $| f(x, t) |$ in $O < t < a$, $| x | \leqslant b$

Further if we put $B(t) = t$ and let $A(t)$ be a solution of (4) with $A(o) = A'(o) = o$ and $\alpha = 1$ we get the results of [2].

Again putting $\Phi(x, t) = | x |$, we obtain the successive approximations theorem of Coddington and Levinson [3].

Note.—For $\alpha = 1$, the above theorems were pointed by [1] but our results are more general in the sense that α takes any positive real value $\alpha \geqslant 1$.

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LATITUDE DEPENDENCE OF AMPLITUDE AND PHASE OF DIURNAL VARIATION

By

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ABSTRACT

The latitude dependence of amplitude and phase of the diurnal variation of meson intensity has been studied by analysing the data of I. G. Y. stations, Gulmarg ($\lambda = 24{\cdot}7°$N), Ottawa ($\lambda = 57°$N), Chruchill ($\lambda = 70°$N) and Resolute ($\lambda = 83°$N) for the year 1958.

INTRODUCTION

For understanding the energy spectrum of the primary anisotropy responsible for the daily variation one of the very important tool, is the study of the latitude dependence of the amplitude and phase of daily variation of cosmic-ray intensity. Dorman has calculated the energy spectrum for the period 1937-51 using the data of meson intensity recorded by ionization chambers. During the recent I. G. Y., Cosmic-ray meson telescopes have been operated at several stations in the world by various research groups. This has provided a unique opportunity for studying the problem of latitude dependence of the diurnal variation of meson intensity.

Thus with this objective in view, we analysed the pressure corrected meson intensity data for I. G. Y. stations Ottawa (57°N), Churchill (70°N) and Resolute (83°N) along with that of Gulmarg (24·7°N) for the period January to August, 1958, to examine the latitude dependence of the diurnal variation.

EXPERIMENTAL RESULTS

The values of the amplitude and phase of the diurnal component of meson intensity, recorded with meson cubical telescope at Gulmarg, Ottawa, Churchill and Resolute, were determined with the help of harmonic analysis using 12 bihourly values. The results obtained are shown in the tables (1, 2) and the figure 1.

TABLE 1

Geomag. Lat. λ	Station	A^D	Year
57°N	Ottawa	0·39 ± 0·008	1958
70°N	Churchill	0·25 ± 0·008	,,
83°N	Resolute	0·06 ± 0·008	,,

TABLE 2

Geomag. Lat. λ	Station	ϕ^D	Year
24·7°N	Gulmarg	212°	1958
57°N	Ottawa	194°	,,
70°N	Churchill	148°	,,
83°N	Resolute	124°	,,

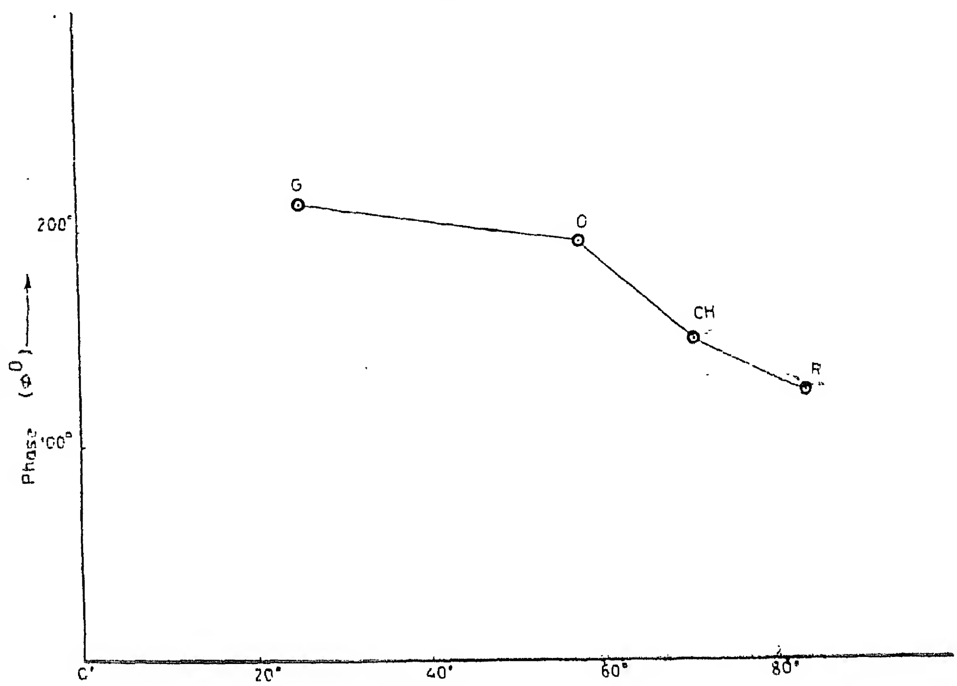

Fig. 1 – Latitude dependence of phase of the diurnal component of pressure corrected meson intensity.

DISCUSSION AND CONCLUSION

Thompson (1938), and Thambyahpillai and Elliot (1953) studied the latitude dependence of the diurnal variation of cosmic-ray intensity, but they found no significant dependence on latitude. Firor et al (1954) have also examined the daily variation of the nucleonic component at two different latitudes viz., Huancayo ($\lambda = 0^\circ$), Climax ($\lambda = 48^\circ$N) and found that the ratio of the peak to peak variation at these two stations is $1{\cdot}4 \pm 0{\cdot}02$. The analysis of the data of Huancayo ($\lambda = 0^\circ$), Christchurch ($\lambda = 48^\circ$S) and Godhavan ($\lambda = 80^\circ$N) where the hard component data have been averaged over eight years, brings out the following facts (Dorman 1957) :

(1) That the variations at Cheltenham and Christchurch which are symmetrically situated with respect to equator are similar and the hour of maximum is also approximately the same.

(2) That the amplitude of variation at Godhavan ($\lambda = 80^\circ$N) which is situated near the pole is smaller than at the other three stations and

(3) that the diurnal variations are apparently of irregular character.

The study of the amplitude of the diurnal variation of the meson intensity recorded with meson cubical telescope at Ottawa ($\lambda = 50^\circ$N), Churchill ($\lambda = 70^\circ$N) and Resolute ($\lambda = 83^\circ$N) reveals that it does not remain constant over the various latitudes considered. It is apparent from the table (1) that the amplitude does not show any systematic variation with latitude. However, we see for the station situated near the pole ($\lambda = 83^\circ$N), the amplitude is appreciably smaller than others.

Leaving Gulmarg (9,000 ft.), the other stations of the table (2) are situated at sea level. Sarabhai et al. (1955) have shown that the phase of the diurnal component is earlier at high altitude for the same latitude. It means that the phase for a station which would be situated at the same latitude as Gulmarg but sea-level will be somewhat greater than what has been shown in the table (2). So our conclusions regarding the latitude dependence of phase, will not be effected because of the difference of altitude of Gulmarg from other stations.

Form table (2) and the figure (1) it appears that the phase ϕ^D of the diurnal component of meson intensity is earlier at higher geomagnetic latitude or we may say that the phase of the diurnal component decreases as latitude in creases.

ACKNOWLEDGEMENT

The author is highly indebted to Prof. P. S. Gill for his kind interest and guidance throughout the work. He is also thankful to the Union Ministry of Education for providing a Senior Research Scholarship and to the staff of Gulmarg research observatory particularly Mr. Lekh Vir Sood for his useful discussions.

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SUPERPOSABILITY AND STREAMLINE PATTERN

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§ 0. ABSTRACT

Kapur [2] has deduced the relations existing between two (*a*) Plane (*b*) Axisymmetrical (Poloidal) Self-superposable and mutually superposable flows. This paper examines the stream patterns for such flows. They are either circles or parallel straight lines, when (*i*) the fluid is non-viscous and iso-vels coincide with stream lines. (*ii*) the flow is viscous and self-superposable. (*iii*) Self-superposable flows which are mutually superposable.

The steady self-superposable poloidal flows, which are mutually svperposable are axial.

§ 1. (A). We shall obtain possible stream patterns for plane flows whose iso-vels and iso-curls coincide with streamlines. i. e.

(1) $q = q(\psi)$ and (2) $\zeta = \zeta(\psi)$

The condition (1) is necessary and sufficient for stream lines to be parallel curves [3]. Also condition (2) is the same when the flow is self-superposable [2] or motion to be steady when the fluid is non-viscous [6]. The curvature of the stream line is

$$\rho^{-1} = \text{Div}\left[\frac{\text{grad}\,\psi}{|\,\text{grad}\,\psi\,|}\right] = q^{-1}(\psi)\,\zeta(\psi) - \frac{dq(\psi)}{a\psi} \tag{3}$$

This shows that the streamlines are concentric circles. Also proceeding as Ghildyal [1] from (3) we get

either $\psi_n = o$ or $\zeta(\psi) = q(\psi)\,\frac{dq(\psi)}{a\psi}$

either way it may be observed that the streamlines are parallel straight lines. Hence we have theorem

I. Theorem: The stream patterns for a class of plane rotational flows whose iso-curls and iso-vels coincide with streamlines are a system of concentric circles or a set of parallel straight lines

(B). We shall examine the nature of the flow whose iso-curls coincide with streamlines and velocity 'q' is constant throughout the region.

i.e. (4) $\quad \zeta = \zeta(\psi)$ and (5) q = constant.

In this case (3) becomes

(6) $\quad \rho^{-1} = \zeta(\psi)\, q^{-1}$

Again proceeding as Ghildyal [1] we have either $\psi_x = o$ or $\zeta = 0$

In either case motion is irrotational. Hence we have

II Theorem : There can be no plane rotational flow, whose velocity 'q' is constant in magnitude throughout the region of flow.

(C). We shall examine the stream patterns for a self-superposable rotational, viscous steady flows. It has to satisfy (2) in addition to integrability condition.

(7) $\quad \nabla^2\zeta(\psi) = \nabla^4\psi = o$ or (8) $\zeta''(\psi)\, q^2 + \zeta'(\psi)\, \zeta(\psi) = o$

(Therefore by theorm II 'q' is constant on ψ . All the conditions of theorem I are satisfied. Hence we have

"All rotational self-superposable flows when the fluid is viscous and motion steady have the stream patterns as stated in Theorem I."

(D). Consider the stream patterns for all rotational self-superposable flows, which are mutually superposable.

Kapur [2] has obtained the conditions for two such flows (ψ, ζ,) and (ψ_2, (ζ_2) as :—

(9) $\quad \zeta_2 = k(\zeta_1)$ $\qquad$ and (10) $\quad \psi_2 = H(\psi_1)$

(11) $\quad \zeta_1 = f_1(\psi)$ $\qquad$ and (12) $\quad \zeta_2 = f_2(\psi_2)$

(α) Stream patterns will be the same as stated in theorem I, when the fluid is viscous and motion steady.

(β) When the fluid is non-viscous.

(13) $\nabla^2 \psi_2 = \zeta_2 = k \{ f_1 (\psi_1) \} = H'' (\psi_1) q_1^2 + H' (\psi_1) f_1 (\psi_1)$

From (13) and theorem II it follows that 'q' must be constant on ψ_1. Again the stream patterns for (ψ_1, ζ_1) are the same as in theorem I. Also from (10) it follows that (ψ_2, ζ_2) will have the same patterns as (ψ_1, ζ_1)

From (α) and (β) we can conclude :—

"The only possible stream patterns for all rotational self-superposable flows which are mutually superposable (whether the fluid is viscous or not) are the same as stated in theorem I."

§ 2. AXISYMMETRICAL (POLOIDAL) FLOWS

(A) We shall consider self-superposable rotational steady flows.

The self-superposability condition [2] is

(14) $\omega^{-1} \zeta = P(\psi)$

If iso-vels coincide with streamlines we have

(15) $\omega \, q(\psi) = (\psi^2{}_x + \psi^2{}_\omega)^{\frac{1}{2}} = |\operatorname{grad} \psi|$

The expression for curvature becomes

(16) $\rho^{-1} = \omega \left\{ P(\psi)^{-1} q(\psi) - \dfrac{dq(\psi)}{d\psi} \right\} = \omega \, H(\psi)$

say which shows that the curvature varies inversely with the distance from the exis of rotation along the stream line. Again we can prove as in §1 that no rotational poloidal flow exist if 'q' is constant throughout the region of flow. We have also proved [5] that if the fluid is viscous motion steady and self-superposable, the flows are axial.

(B) We shall determine the stream patterns for all rotational self-superposable axisymmetrical (Polodial) flows which are mutually superposable.

Kapur [2] has obtained the following conditions for two such poloidal flows (ψ_1, ζ_1) and (ψ_2, ζ_2) to be mutually superposable as

(17) $\psi_2 = G(\psi_1)$ and (18) $\omega^{-1} \zeta_2 = H(\omega^{-1} \zeta_1)$

(19) $\omega^{-1} \zeta_1 = f_1 (\psi_1)$ and (20) $\omega^{-1} \zeta_2 = f_2 (\psi_2)$

(α) If the fluid is viscous, they are axial by the result stated above.

(β) If the fluid is non-viscous, operating E^2 on (17) and using (18), (19) and (20) we have

(21) $\quad H\{f_1(\psi_1)\} = G''(\psi_1)\, q^2_1 + G^1(\psi_1) f_1(\psi_1)$

Hence q^2 is constant only on ψ_1 = constant. Also

(22) $\quad p_1 \rho_1^{-1} = -\frac{1}{2}\int d^2 q_1 + \int \omega^{-1}\zeta_1\, d\psi_1 +$ constant, which is a function of ψ_1 i.e. velocity, pressure and $\omega^{-1}\zeta$, are constant along individual streamlines. for the first flow which according to Prim [4] is axial.

From (17) it follows that the flow (ψ_2, ζ_2) is also axial.

Hence we have

III Theorem :—All rotational self-superposable steady poloidal flows which are mutually superposable (whether the fluid is viscous or non-viscous) are axial.

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KINETICS OF OXIDATION OF MERCUROUS NITRATE BY POTASSIUM PERSULPHATE IN AQUEOUS SOLUTION. PART I — UNCATALYSED REACTION

By

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ABSTRACT

The kinetics of oxidation of mercurous nitrate in dilute nitric acid solution by potassium persulphate has been investigated in aqueous medium. The results are fairly reproducible when the necessary precautions about the purity of the materials and the medium are taken. The over-all order of the reaction is one, being first order with respect to persulphate and zero order with respect to mercurous ions. The reaction is attended with an induction period, the extent of which increases with increasing concentration of mercurous nitrate and decreasing with increasing concentration of persulphate. The temperature coefficient for the reaction comes out to be 3·1685 for 0·01 M concentration of the reactants. From this the energy of activation, frequency factor and the entropy of activation have been calculated. There is no appreciable effect of HNO_3 on the rate of this reaction.

INTRODUCTION

During the systematic study of the oxidation of different reducing cations by persulphate ions, it was observed by us that mercurous ion is slowly oxidised by persulphate ion. Much work has been carried out on the kinetics of silver catalysed redox reactions involving persulphate ion. However the uncatalysed reactions involving persulphate ion have not received much attention so far.

The present paper deals with the kinetic study of the hitherto unstudied oxidation of mercurous nitrate by potassium persulphate in the absence of the catalyst. This study has been undertaken to find out if this reaction follows a similar behaviour as other uncatalysed reactions of persulphate ion or not.

EXPERIMENTAL

Potassium persulphate G. R., E. Merck, mercurous nitrate E. P., E. Merck, nitric acid A. R., Basynth, have been used. All other chemicals used were of A. R., B. D. H. quality. Potassium iodide was of E. P., E. Merck quality.

The standard solution of potassium persulphate was prepared fresh daily by dissolving the required amount of the salt in redistilled water and its strength was checked by the iodometric method of Szabo, Csyani and Galiba[1], as modified by Khulbe and Srivastava[2]. The same method was used for studying the progress of of the reaction (as described below).

The solution of mercurous nitrate was prepared in the minimum concentration of dilute nitric acid to prevent hydrolysis. The strength of mercurous nitrate solution was determined by titrating with standard KBr solution using bromophenol blue as adsorption indicator, according to the method proposed by Zombory and modified by Kolthoff and Larson[3]. A glass stoppered Pyrex conical flask blackened outside with black japan and wrapped in a black cloth was used as the reaction vessel. The progress of the reaction was studied as follows : 5 ml. of the reaction mixture was pipetted out at different intervals of time and added to 5 ml. of a 10%

solution of sodium chloride. The mixture was centrifuged for about 10-15 minutes to remove the precipitated mercurous chloride and then 5 ml of the clear liquid was pipetted out from the centrifuge tube and added to 1 ml. of a mixture of 1% $CuSO_4$ and 1% $FeSO_4$ solution in an iodine flask, and then a 15% solution of KI was added The flask was kept for about 5-6 minutes for the complete liberation of iodine and titrated against a standardised sodium thiosulphate solution. The blank reading with 1 ml. $CuSO_4$ and $FeSO_4$ mixture was taken which was subtracted from the above titre value to obtain the amount of potassium persulphate remaining in terms of titre value of sodium thiosulphate.

Preliminary experiments showed that unlike $Sn^{++} - S_2O_8''$ reaction studied by Gupta and Srivastava[4] earlier the uncatalysed reaction is a slow reaction and has a measurable velocity at 40°C. and above and at 0·01 M or higher concentration of potassium persulphate. Hence the measurements were carried out at 50°C., unless otherwise stated.

RESULTS OF THE MEASUREMENTS

First of all the reaction was repeated several times at different equimolecular concentration of the reactants to see if the results are reproducible or not. It was found that the results are fairly reproducible when necessary precautions about the extreme purity of the materials and the medium are taken. The following table gives the results in two typical runs each at equimolecular concentration of the reactants.

TABLE 1

$K_2S_2O_8 = Hg_2(NO_3)_2 = 0{\cdot}01$ M ; $HNO_3 = 0{\cdot}154$ N ; Temperature = 50°C.
Initial reading = 5·02 ml.

Times in minutes	Vol. of N/100 hypo used		k unimolecular $\times 10^3$
	Expt. I	Expt. II	
10	4·86	4·91	
20	4·72	4·78	2·695
30	4·44	4·49	4·479
45	3·95	3·98	6·001
60	3·52	3·51	6·716
90	2·76	2·77	7·158
120	2·11	2·15	7·511
150	1·63	1·63	7·877
180	1·36	1·37	7·506
210	1·07	1·07	7·621

The above data shows the reproducible nature of the results obained. Further it is found that the reaction has a fairly long induction period. After the induction period is over, there is a fair constancy in the unimolecular rate constant values,

This leads one to conclude that the reaction follows an unimolecular behaviour and the over-all order of the reaction is one at equimolecular concentration of the reactants

Further, the reaction was carried out at other equimolecular concentrations of the reactants to verify the unimolecularity of the reaction. The results for three different equimolecular concentrations of the reactants are represented in fig. 1,

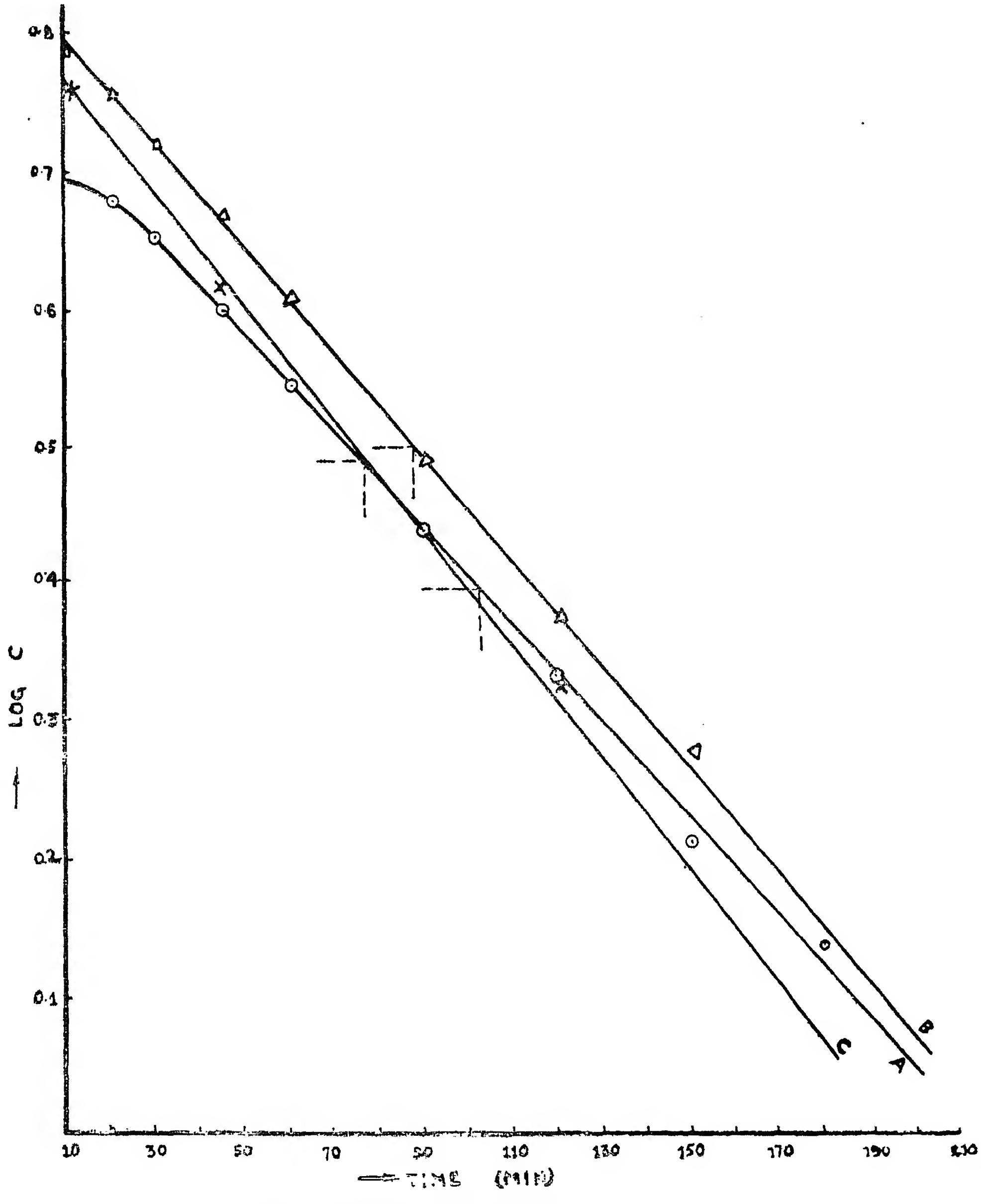

Fig. 1 : Variation of log c with time at 50°C

	A	B	C
Concs : $K_2S_2O_8$	0·01 M	0·025 M	0·05 M
$Hg_2(NO_3)_2$	0·01 M	0·025 M	0·05 M

where log c (c in terms of titre values in ml.) is plotted against time. The curves are nearly linear showing the unimolecularity of the reaction. Further it is seen that the induction period almost disappears at higher concentration of the reactants.

EFFECT OF POTASSIUM PERSULPHATE CONCENTRATION

In order to determine the effect of $K_2S_2O_8$ concentration on rate and thereby to determine the order of the reaction with respect to S_2O_8'', the reaction was carried out at three different initial concentrations of $K_2S_2O_8$ and at constant concentration of $Hg_2(NO_3)_2$, the results of which are represented in fig. 2. The curve of

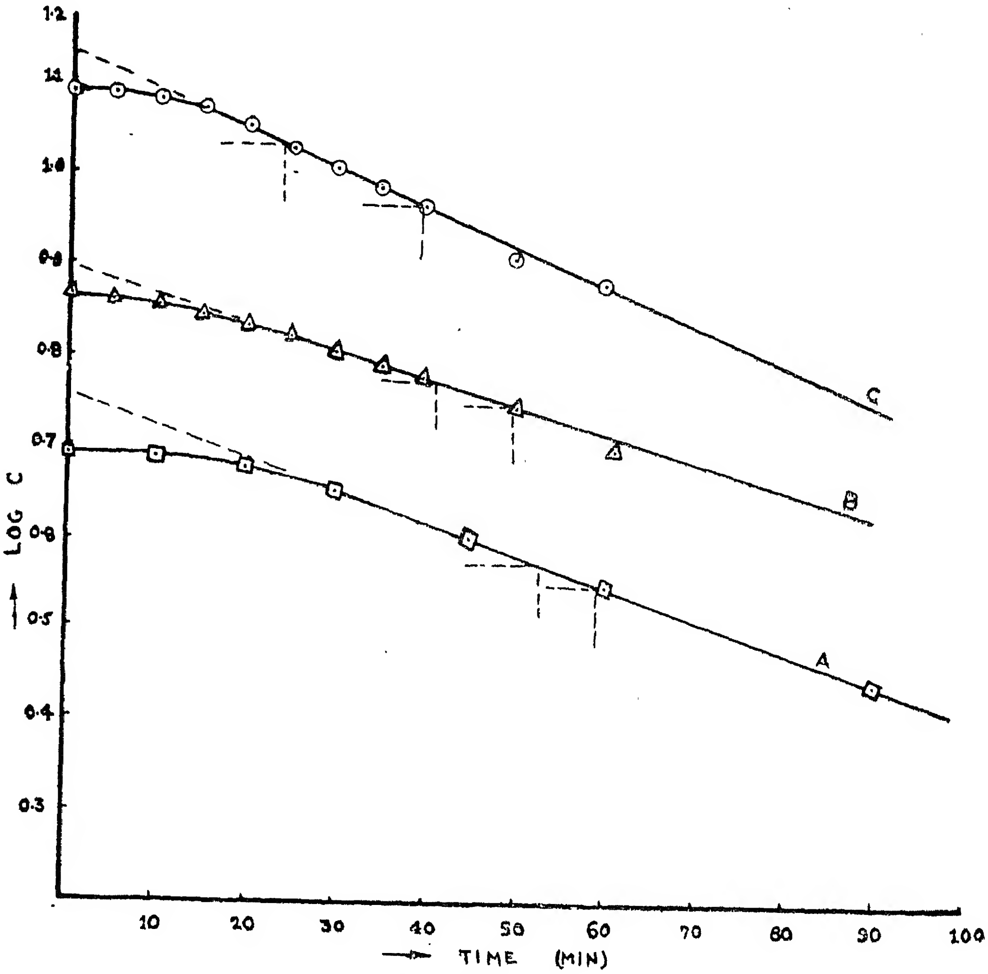

Fig. 2 : Variation of log c with time at 50°C

Conc. of $Hg_2(NO_3)_2$: 0·01 M

Conc. of $K_2S_2O_8$: A 0·01 M ; B 0·015 M ; C 0·025 M

$\log c$ vs. t in each case is linear showing the unimolecularity of the reaction. Further it is seen that the extent of the induction period decreases on increasing the concentration of potassium persulphate. The value of the specific rate constant as extrapolated from the curves comes out to be 8.093×10^{-3} min^{-1}, $7{\cdot}108 \times 10^{-3}$ min^{-1} and $10{\cdot}02 \times 10^{-3}$ min^{-1} for 0·01 M, 0·015 M and 0·025 M potassium persulphate respectively.

When the time for the same amount of potassium persulphate to decompose (corresponding to 1·5 ml. of hypo solution) at the three concentrations of potassium persulphate 0·01 M, 0·015 M and 0·025 M is extrapolated, it comes out to be 59, 41 and 23·5 minutes respectively. It shows that the rate depends on the potassium persulphate concentration.

The time taken for the same fraction of $K_2S_2O_8$ (1/4th.) to decompose as read out from the curves, is 53, 49·5 and 39·25 minutes for 0·01 M, 0·015 M and 0 025 M $K_2S_2O_8$ respectively which suggests that the order of the reaction is higher than one. However, this may be due to the fact that the extent of induction period depends on $K_2S_2O_8$ concentration.

To confirm the order of the reaction with respect to S_2O_8'' ion the differential method has been applied and for this the values of c (in terms of titre values) have been plotted against t (time in minutes) in fig. 3 and from the curves thus obtained the values of $-dc/dt$ for 0·01 M, 0·015 M and 0·025 M initial concentration of $K_2S_2O_8$ have been extrapolated at 45 minutes (after the end of the induction period). On substitution of these values of $-dc/dt$ in the expression

$$n = \frac{\log(-dc_1/dt) - \log(dc_2/dt)}{\log c_1 - \log c_2}$$

the value of n comes out to be 0·97 and 1·05, confirming the unimolecularity of the reaction with respect to S_2O_8''.

EFFECT OF MERCUROUS NITRATE CONCENTRATION

The reaction was carried out at four different initial concentrations of mercurous nitrate, keeping the concentration of potassium persulphate constant. The

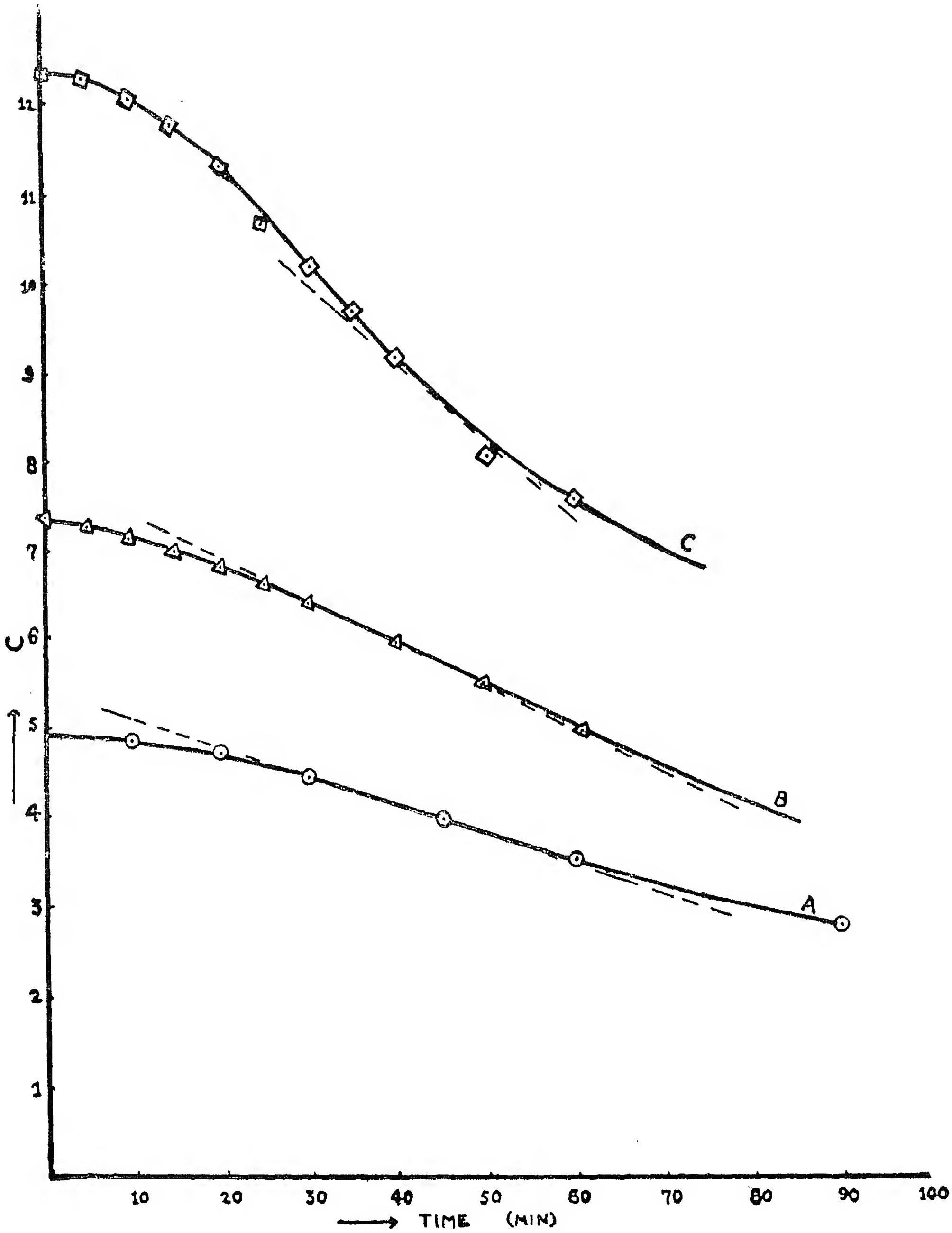

Fig. 3 : Variation of c with time at 50°C
Conc. of $Hg_2(NO_3)_2$: 0·01 M
Conc. of $K_2S_2O_8$: A 0·01 M ; B 0·015 M ; C 0·025 M

results of these experiments have been plotted in fig. 4, where the titre values after conversion into mercurous nitrate concentration have been plotted against time. From these curves the time taken for the same fraction (1/4th) of $Hg_2(NO_3)_2$ to

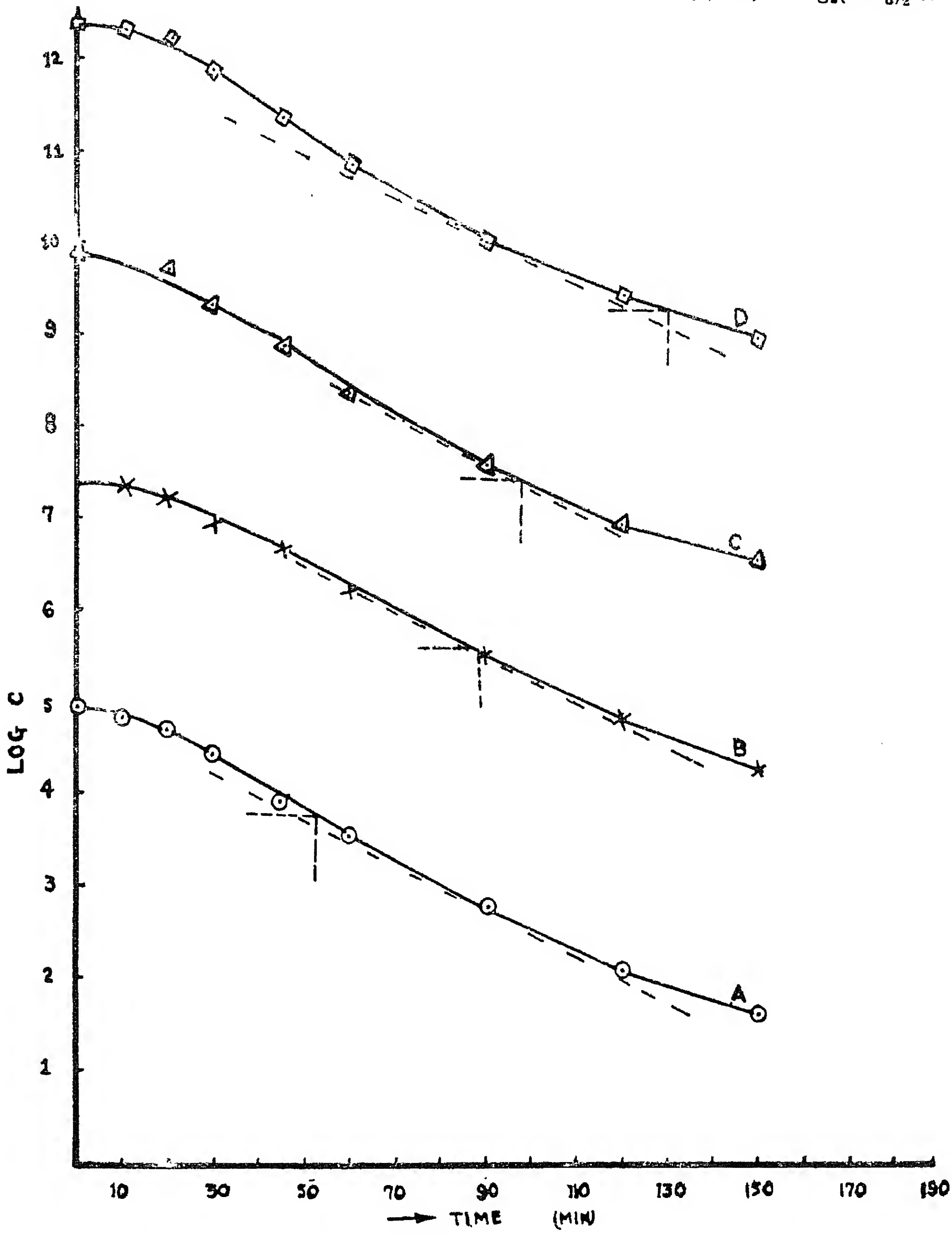

Fig. 4 : Variation of c with time at 50°C
Conc. of $K_2S_2O_8$: 0·01 M
Conc. of $Hg_2(NO_3)_2$: A 0·01 M : B 0·015 M ; C 0·02 M ; D 0·025 M

oxidise comes out to be 53, 88·5, 98 and 130 minutes for 0·01 M, 0·015 M, 0·02 M and 0·025 M $Hg_2(NO_3)_2$ concentration respectively. On substitution of these values in the expression

$$n = 1 + \frac{\log t_1/t_2}{\log c_2/c_1}$$

the value of n obtained is -0·26, 0·11 and 0·02, showing thereby that the reaction is zero order with respect to Hg_2^{++}. The value of n by the differential method from these curves also comes out to be -0·002, 0·10 and -0·02 confirming the zero molecularity with respect to Hg_2^{++}. Further it is seen that the extent of induction period increases with an increase in $Hg_2(NO_3)_2$ concentration. This suggests that probably some thing present in mercurous nitrate solution is responsible for the induction period.

EFFECT OF TEMPERATURE

In order to determine the temperature coefficient and thereby the energy of activation, the reaction was carried out at four different temperature from 40°C. to 55°C., the results for which are represented in fig. 5 where log c (in terms of the

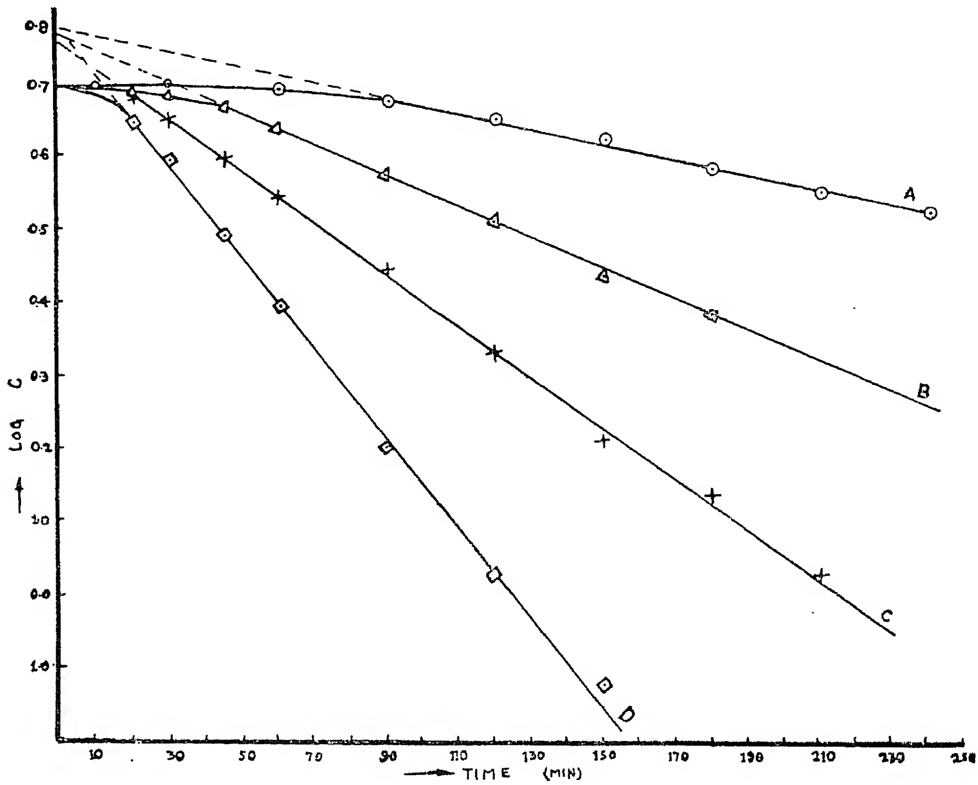

Fig. 5 : Variation of log c with time
Conc. : $K_2S_2O_8 = Hg_2(NO_3)_2 = 0{\cdot}01$ M
Temperature : A 40°C ; B 45°C ; C 50°C ; D 55°C

titre values) is plotted against time. From the curves at these four temperatures the the value of specific rate has been evaluated and thereby the value of temperature coefficient, energy of activation. frequency factor and entropy of activation have been calculated. These values have been tabulated in the following table.

TABLE 2

$K_2S_2O_8 = Hg_2(NO_3)_2 = 0{\cdot}01$ M ; $HNO_3 = 0{\cdot}1540$ N

Temperature in °K	k unimolecular $\times 10^3$	Temperature coefficient	Energy of activation	Frequency factor in sec.$^{-1}$	Entropy of activation in E. U.
313	2·413	...	...	1×10^{12}	−3·727
		3·353	24·32 Cals.		
323	8·091	...	...	$1{\cdot}047 \times 10^{12}$	−3·698
318	4·826	...	...		
		2·984	22·68 Cals.	$1{\cdot}122 \times 10^{12}$	−3·530
328	14·400	...	...	$1{\cdot}072 \times 10^{12}$	−3·683
Mean values ...		3·1685	23·5 Cals.	$1{\cdot}06 \times 10^{12}$	− 3·659

The reaction has a very high temperature coefficient and the curves further show that the extent of induction period at higher temperatures is less than that at lower temperatures.

EFFECT OF NITRIC ACID

In order to decide whether the concentration of HNO_3 affects the rate, the reaction was carried out at three different initial concentrations of HNO_3, keeping the concentration of the reactants the same in each case. It was seen that there is no appreciable effect of nitric acid concentration on the rate as is evident from the following data.

TABLE 3

$K_2S_2O_8 = Hg_2(NO_3)_2 = 0{\cdot}01$ M ; Temperature = 50°C.

Concentration of HNO_3	k unimolecular $\times 10^3$ from the curves
0·1540 N	7·598 min^{-1}
0·4780 N	7·278 min^{-1}
0·8020 N	7·015 min^{-1}

The specific rate is found to decrease by hardiy 1% for a more than five fold increase in the concentration of HNO_3.

CONCLUSION

Thus it is seen that this uncatalysed oxidation of Hg_2^{++} with S_2O_8'' is similar in behaviour to other uncatalysed reactions of potassium persulphate where the reaction is found to be unimolecular with respect to S_2O_8'' and zero molecular with respect to the reductant.

Further work to elucidate the exact mechanism of the reaction. to determine the cause of induction period and the effect of catalytic ions is in progress.

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ON THE GENERALISED LAPLACE TRANSFORM

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ABSTRACT

Meijer gave two generalisations of the classical Laplace transform in the form

$$\phi(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{pt}\; K_m(pt) f(t)\, dt \tag{1.1}$$

and $$\phi(p) = p \int_0^\infty (pt)^{-k-\frac{1}{2}} e^{-\frac{1}{2}pt}\, W_{k+\frac{1}{2}, m}(pt) f(t)\, dt. \tag{1.2}$$

In the present paper some of the properties of the transform (1.1) have been given, which are useful in evaluating infinite integrals. One property of the transform represented by (1.2) has also been established at the end.

1. Meijer (1940) gave a generalisation of the well-known Laplace transform, viz.

$$\phi(p) = p \int_0^\infty e^{-pt} f(t)\, dt, \quad R(p) > 0, \tag{1.1}$$

by means of the relation given by

$$\phi(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty (pt)^{\frac{1}{2}}\, k_m(pt) f(t)\, dt, \tag{1.2}$$

He again generalised the Laplace transform in (1941) in the form of another transform now called the Meijer transform given by

$$\phi(p) = p \int_0^\infty (pt)^{-k-\frac{1}{2}} e^{-\frac{1}{2}pt}\, W_{k+\frac{1}{2}, m}(pt) f(t)\, dt, \tag{1.3}$$

For $k = \pm m$, (1.1) reduces to the Laplace transform,

$$\phi(p) = p \int_0^\infty e^{-pt}\, f(t)\, dt.$$

Varma has generalised the Laplace transform, in the form

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{m-\frac{1}{2}} W_{k,m}(pt) f(t)\, dt, \tag{1.4}$$

Since $W_{-m+\frac{1}{2}, m}(x) \equiv x^{-m+\frac{1}{2}} e^{-\frac{1}{2}x}$, (1.4) reduces to (1.1) when $k = -m + \frac{1}{2}$.

The object of this note is to investigate some of the properties of the generalised Laplace transform given in the form (1.2). At the end I have given a property of the transform given by (1.4).

2. *Theorem* :—If $\phi(s)$ is the generalised Laplace transform of $g(x)$ and $g(x)$ is the cosine transform of $f(x)$, then

$$\phi(s) = \int_0^\infty f(y)\, \psi(y)\, dy,$$

where $\psi(y) = \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!} s^{-(2r+\frac{3}{2})} 2^{2r-\frac{1}{2}} \Gamma_x \left(\gamma + \frac{3}{4} \pm \frac{m}{2} \right) y^{2r}$.

provided that the integral exists.

Proof :—Since $\phi(s)$ is the generalised Laplace transform of $g(x)$, it will be given by

$$\phi(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty (sx)^{\frac{1}{2}} k_m(sx)\, g(x)\, dx.$$

Also, $g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy) f(y) dy$

Therefore, $\phi(s) = \frac{2}{\pi} \int_0^\infty (sx)^{\frac{1}{2}} k_m(sx)\, dx \int_0^\infty \cos(xy) f(y)\, dy.$

Changing the order of integration, which is permissible, we get

$$\phi(s) = \frac{2}{\pi} \sqrt{s} \int_0^\infty f(y)\, dy \int_0^\infty (sx)^{\frac{1}{2}} \cos(xy)\, k_m(sx)\, dx,$$

$$= \frac{2}{\pi} \sqrt{s} \int_0^\infty f(y)\, dy \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!} y^{2r} \int_0^\infty x^{2r+\frac{1}{2}} k_m(sx)\, dx.$$

Note : $\Gamma_x(a \pm b) = \Gamma(a+b)\, \Gamma(a-b)$

Evaluating the integral, with the help of the known integral, *viz.*

$$\int_0^\infty x^{\mu-1}\, k_m(x)\, dx = 2^{\mu-2}\, \Gamma_x\left(\frac{\mu}{2} \pm \frac{m}{2}\right), \quad (\mu \pm m) > 0.$$

we get $\phi(s) = \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\, s^{-(2r+1)}\, 2^{2r-\frac{1}{2}}\, \Gamma_x\left(r + \frac{3}{4} \pm \frac{m}{2}\right) \times$

$$\times \int_0^\infty y^2\, f(y)\, dy = \int_0^\infty \psi(y)\, f(y)\, dy,$$

where $\chi(y) = \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\, s^{-(2r+\frac{3}{2})}\, 2^{2r-\frac{1}{2}}\, \Gamma_x\left(r + \frac{3}{4} \pm \frac{m}{2}\right) y^{2v}.$

Thus the theorem is established.

The change in the order of integration will be justified if

(*i*) $\int_0^\infty f(x)\, k_m(x)\, dx$

and (*ii*) $\int_0^\infty \sqrt{x}\, k_m(x)\cos(xy)\, dx$

are both absolutely convergent. Assuming (1) to be absolutely convergent, (*ii*) will converge absolutely if $\left(r + \frac{3}{4} \pm \frac{m}{2}\right) > 0$,

Example 1. Consider the relation

$$\int_0^\infty e^{-x^2}\cos(xy)\, dx = \sqrt{\frac{\pi}{2}}\, e^{-y^2/4}$$

We have, $\int_0^\infty f(y)\, \chi(y)\, dy$

$$= \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\, s^{-(2r+3/2)}\, 2^{2r-\frac{1}{2}}\, \Gamma_x\left(r + \frac{3}{4} \pm \frac{m}{2}\right) \int_0^\infty e^{-y^2}\, y^{2r}\, dy.$$

Evaluating this, we have

$$\int_0^\infty f(y)\, \chi(y)\, dy = \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}\, s^{-(2r+3/2)}\, \Gamma_x\left(r + \frac{3}{4} \pm \frac{m}{2}\right) 2^{2r-3/2}\, \Gamma(r+\tfrac{1}{2})$$

Example 2. We know, (Erdelyi 307—4), that

$$\int_0^\infty \cos(st)\, t^{v-1}\, \mathrm{Erfc}(at)\, dt$$

$$= \frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}{\nu\pi\ a^{\nu}}\ {}_2F_2\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{1}{2},\frac{\nu}{2}+1;\frac{-S^2}{4a^2}\right),$$

$[R(v)>0, R(a)>0.]$

Now, $\int_0^\infty f(t)\,\chi(t)\,dt$

$$= \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!}S^{-(2r+3/2)}\,2^{2r-\frac{1}{2}}\,\Gamma_x\left(r+\frac{3}{4}\pm m\right)\int_0^\infty y^{\nu+2r-1}\,\mathrm{Erfc.}(ay)\,dy$$

On using the result (Erdelyi 306-1),

$$\int_0^\infty x^{\rho-1}\,\mathrm{Erfc}\,(ax)\,dx = \frac{\Gamma\left(\frac{\rho}{2}+\frac{1}{2}\right)}{\pi\rho\ a^{\rho}},\quad [R(\rho)>0, R(a)>0.]$$

we get

$$\int_0^\infty f(t)\,\chi(t)\,dt = \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!}S^{-(2r+3/2)}\,2^{2r-\frac{1}{2}}\,\frac{\Gamma\left(r+\frac{v}{2}+\frac{1}{2}\right)\Gamma_x\left(r+\frac{3}{4}\pm\frac{m}{2}\right)}{\pi(2r+v)\ a^{2r+v}}$$

3. *Theorem* 2. If $\phi(s)$ is the generalised Laplace Transform of $g(x)$ and $g(x)$ is the sine transform of $f(x)$, then

$$\phi(s) = \int_0^\infty f(y)\,\chi(y)\,dy,$$

where $\chi(y) = \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)!}\,\frac{y^{2r+1}}{S^{2r+3/2}}\,2^{2r+\frac{1}{2}}\,\Gamma_x\left(r+\frac{5}{4}\pm\frac{m}{2}\right).$

The proof is after the manner of the above theorem.

4. *Theorem* 3. If $\phi(s)$ be the generalised Laplace transform of $g(x)$ and $g(x)$ is the Humbert transform of $f(x)$, then

$$\phi(s) = \int_0^\infty \chi(y)\,f(y)\,dy,$$

where

$$\chi(y) = \sqrt{\frac{S}{\pi}}\sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma_x\left(\frac{v}{2}+r+\frac{5}{4}\pm m\right)}{\Gamma(r+3/2)\,\Gamma(r+\nu+3/2)}\,y^{v+2r+3/2}\,S^{-(v+2r+5/2)}$$

Proof :—Since $\phi(s)$ is the generalised Laplace transform it will be given by

$$\phi(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty (sx)^{\frac{1}{2}}\,k_m(sx)\,g(x)\,dx,$$

Also, $g(x) = \int_0^\infty \sqrt{xy}\; H_v(xy) f(y)\, dy.$

Therefore, $\phi(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty (sx)^{\frac{1}{2}} k_m(sx)\, dx \int_0^\infty \sqrt{xy}\; H_v(xy) f(y)\, dy.$

Changing the order of integration, which is permissible, we get

$$\phi(s) = \sqrt{\frac{2s}{\pi}} \int_0^\infty \sqrt{y}\; f(y)\, dy \int_0^\infty \sqrt{x}\; H_v(xy)\, k_m(sx)\, dx,$$

$$= \sqrt{\frac{2s}{\pi}} \int_0^\infty \sqrt{y}\; f(y)\, dy \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{y}{2}\right)^{v+2r+1}}{\Gamma(r+\frac{3}{2})\; \Gamma(r+v+\frac{3}{2})} \int_0^\infty x^{v+2r+3/2}\; k_m(sx)\, dx,$$

$$= \sqrt{\frac{S}{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r\; \Gamma_x\left(\frac{v}{2}+r+\frac{5}{4}\pm\frac{m}{2}\right)}{\Gamma(r+3/2)\; \Gamma(r+v+3/2)\; S^{v+2r+5/2}} \int_0^\infty y^{v+2r+\frac{3}{2}} f(y)\, dy,$$

$$= \int_0^\infty \chi(y) f(y)\, dy,$$

where $\chi(y) = \sqrt{\frac{S}{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r\; \Gamma_x\left(\frac{v}{2}+r\pm\frac{m}{2}\frac{5}{4}\right)\; y^{v+2r+3/2}}{\Gamma(r+3/2)\; \Gamma(r+v+3/2)\; S^{v+2r+5/2}}.$

Thus the theorem is established.

The change in the order of integration will be justifiable, if the integrals

(i) $\int_0^\infty \sqrt{x}\; f(x)\, k_m(sx)\, dx$

and

(ii) $\int_0^\infty \sqrt{x}\; H_v(xy, k_m(sx)\, dx$

are absolutely convergent. Assuming (*i*) to be absolutely convergent, the second converges absolutely as term by term integration is permissible.

Example 1. Consider the relation (Erdelyi 161-19)

$$\int_0^\infty x^{m-\frac{1}{2}}\; e^{-ax}\; H_v(xy) \sqrt{xy}\; dx$$

$$= \frac{y^{v+3/2}\; \Gamma(v+m+2)}{2^v\; a^{m+v+2} \sqrt{\pi}\; \Gamma(v+\frac{3}{2})}\; {}_3F_2\left(1, \frac{m+v}{2}+1, \frac{m+v+3}{2}; \frac{3}{2}, v+\frac{3}{2}; \frac{-y^2}{a^2}\right).$$

where $R(a) > 0$, $R(m+v) > -2$.

Now, $\int_0^\infty f(y)\,\chi(y)\,dy.$

$$= \sqrt{\frac{S}{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r \, \Gamma_x\left(\frac{v}{2}+r+\frac{5}{4}\pm\frac{m}{2}\right)}{\Gamma(r+3/2)\,\Gamma(r+v+3/2)} S^{-(v+2r+5/2)} \int_0^\infty e^{-ay}\, y^{m+v+1+2r}\,dy,$$

$$= \sqrt{\frac{S}{\pi}} \sum_{r=0}^{\infty} \frac{(-1)^r \, \Gamma_x\left(\frac{v}{2}+r+\frac{5}{4}\pm\frac{m}{2}\right) S^{-(v+2r+5/2)}\, \Gamma(m+v+2+2r)}{\Gamma(r+3/2)\,\Gamma(r+v+3/2)\, a^{m+v+3/2+2r}}$$

5. *Theorem* 4. If $\phi(s)$ be the generalised Laplace transform of $f(x)$, then

$$\int_0^\infty \frac{\phi(s)}{s^p}\,ds = 2^{-p}\,\frac{\Gamma_x\left(\frac{3}{4}-\frac{p}{2}\pm\frac{m}{2}\right)}{\sqrt{\pi}} \int_0^\infty \frac{f(x)\,dx}{x^{1-p}},$$

provided that the integrals involved are convergent.

Proof :—We know that $\phi(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{Sx}\; k_m(sx)\, f(x)\,dx,$

Therefore, $\int_0^\infty \frac{\phi(s)}{s^p}\,ds = \int_0^\infty \frac{ds}{s^p} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty (sx)^{\frac{1}{2}}\, k_m(sx) f(x)\,dx.$

Changing the order of integration, which is permissible, we get

$$\int_0^\infty \frac{\phi(s)}{s^p}\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{\frac{1}{2}} f(x)\,dx \int_0^\infty S^{\frac{1}{2}-p}\, k_m(sx)\,dx.$$

On evaluating the integral by a well-known formula, we get

$$\int_0^\infty \frac{\phi(s)}{s^p}\,dx = \sqrt{\frac{2}{\pi}} \cdot 2^{-p-\frac{1}{2}}\, \Gamma_x\left(\frac{3}{4}-\frac{p}{2}\pm\frac{m}{2}\right) \int_0^\infty \frac{f(x)dx}{x^{1-p}},\; \left(\frac{3}{2}-p\pm m\right) > 0.$$

$$= \frac{2^{-p}\, \Gamma_x\left(\frac{3}{4}-\frac{p}{2}\pm\frac{m}{2}\right)}{\sqrt{\pi}} \int_0^\infty \frac{f(x)\,dx}{x^{1-p}}.$$

Thus the result is established

The change in the order of integration will be justifiable if both the integrals

$$(i)\ \int_0^\infty \sqrt{x}\ \ f(x)\, k_m(sx)\, dx$$

and $$(ii)\ \int_0^\infty s^{\frac{1}{2}-p}\ \ k_m(sx)\, dx,$$

converge absolutely. Assuming (*i*) to be absolutely convergent, (*ii*) converges absolutely if $\left(\frac{3}{2} - p \pm \frac{m}{2}\right) > 0.$

Particular Case : (1) When $p = \frac{1}{2}$, we have

$$\int_0^\infty \frac{\phi(s)}{\sqrt{s^p}}\, ds = \frac{1}{\sqrt{2\pi}}\ \Gamma_x\left(\frac{7}{2} \pm \frac{m}{2}\right) \int_0^\infty \frac{f(x)}{\sqrt{x}}\, dx.$$

6. *Theorem* 5 :—If $\phi(s)$ be the generalised Laplace transform (given by Varma) of $f(x)$ then

$$\int_0^\infty \frac{\phi(s)}{s^p}\, ds = \frac{\Gamma_x(m-p+2\pm m)}{\Gamma(m-p-k+5/2)} \int_0^\infty x^{p-2}\, f(x)\, dx,$$

provided that the integrals involved exist.

Proof fallows from theorem 4 given above

Particular Cases :—

(*i*) When $p = 1$, we get

$$\int_0^\infty \frac{\phi(s)}{s}\, dx = \frac{\Gamma_x(m+1\pm m)}{\Gamma(m-k+3/2)} \int_0^\infty \frac{f(x)\, dx}{x},$$

(*ii*) When $k = -m + \frac{1}{2}$, then

$$\int_0^\infty \frac{\phi(s)}{s^p}\, ds = \Gamma(2-p) \int_0^\infty \frac{f(x)\, dx}{x^{2-p}},$$

where $\phi(s)$ and $f(x)$ are the Laplace transforms of each other,

(*iii*) When $k = -m + \frac{1}{2}$ and $p = 1$, we get

$$\int_0^\infty \frac{\phi(s)}{s}\, ds = \int_0^\infty \frac{f(x)}{x}\, dx$$

Example —Let $\phi(s) = 2a^{k-\frac{1}{2}} s^{m+1} \quad k_{2m} \;(2\sqrt{as})$

and $f(x) = e^{-a/x} \; x^{k-m-3/2}$,

Then, by the above theorem, we have

$$2a^{k-\frac{1}{2}} \int_0^\infty \frac{s^{m+1}\, k_{2m}\,(2\sqrt{as})}{s^p}\, ds$$

$$= \frac{\Gamma(2-p)\,\Gamma(2-p+2m)}{\Gamma(m-p-k+5/2)} \int_0^\infty \frac{e^{-a/x}\; x^{m+k-3/2}}{x^{2-p}}\, dx$$

On evaluating the integral on the left hand side, we get

$$2a^{k-\frac{1}{2}} \int_0^\infty s^{m-p+1}\; k_{2m}\;(2\sqrt{as})\; ds = a^{k+p-m-2} \quad \Gamma_x\;(m-p+2\pm m).$$

The right hand side integral

$$\int_0^\infty e^{-a/x}\;\; x^{k+p-m-7/2}\, dx = a^{k-p-m-2} \quad \Gamma_x\;(m+\frac{5}{2}-k-p).$$

Thus the left hand side = right hand side, and the theorem is justified.

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NOTE ON THE BOUNDS ON THE NORM OF A SOLUTION OF A SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS

By

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ABSTRACT

The author has discussed the asymptotic behaviour and obtained criteria for the uniqueness and stability of solutions of a system of differential equations using the notion of maximal and minimal solutions of a scalar equation. Continuing this approach, in this note, the author obtains the bounds on the norm of a solution of such a system.

In this note we obtain upper and lower bounds on the norm of a solution of a system of non-linear differential equations. The results are essentially analogous to [2], however, the method of approach is entirely different and depends on a lemma established by the author in [1].

1. We consider the system

$$\frac{dx}{dt} = f(t, x) \tag{1.1}$$

where x denotes a n-dimensional vector and $f(t, x)$ is a given vector field which is defined in the product space $\Delta = I \times R^n$ and is continuous there, where I is the interval $0 \leqslant t \leqslant \infty$ and R^n is Euclidean space of n-dimensions. We now state the lemma proved in [1] in a suitably modified form so as to suit our requirements in § 2.

Lemma I :

Let $V(t, x) \geqslant 0$ be a scalar function also defined on Δ. We assume that V satisfies a Lipschitz condition in x locally. We define

$$V^*(t, x) = \limsup_{h \to 0+} \frac{1}{h}\left[V(t+h, x) - V(t, x)\right]$$

Let $\omega(t, r)$ be measurable in t for fixed r and continuous in r for fixed t. Suppose

$V^*(t, x) \leqslant \omega(t, V(t, x)) - \alpha V(t, x)$

where $\alpha > o$ is a suitably chosen number.

Further let

$| f(t, x) | \leqslant V(t, x)$

where $| x |$ indicates the Euclidean norm of x.

Let $r(t)$ be the maximal solution of

$r' = \omega(t, r)$

with $r(t_0) = r_0, r_0 > o$. Then if $x(t)$ is any solution of (1.1) such that

$V(t, x(t_0) \leqslant r_0$, we have

$$V(t, x(t)) \leqslant r(t) \text{ for } t \geqslant t_0 \tag{1.2}$$

We shall also require the following lemma, which is essentially an extension of iemma I, to prove the Theorem 2 in § 2.

Lemma II :

Let $V(t, x) \geqslant o$ be a scalar function defined on $\triangle$.
we assume that V satisfies the inequality

$$V(t, x_1) - V(t, x_2) \geqslant - L | x_1 - x_2 | \tag{1.3}$$

where $L > o$ is a constant. Fu ther let

$$| f(t, x) | \leqslant V(t, x) \tag{1.4}$$

As in lemma I, we define $V^*(t, x)$ and $\omega(t, \rho)$.

Suppose

$$V^*(t, x) \geqslant - \omega(t, V(t, x)) + \beta V(t, x) \tag{1.5}$$

where $\beta > o$ is a suitably ehosen constant. Then

whenever $o \leqslant \rho_0 \leqslant V(t_0\, x(t_0))$, we have

$$\rho(t) \leqslant V(t, x(t)) \text{ for } t \geqslant t_0 \tag{1.6}$$

where $\rho(t)$ is the minimal solution of

$\rho' = - \omega(t, \rho)$ with $\rho(t_0) = \rho_0, \rho_0 > o$

The proof is quite similar to lemma I and we shall merely indicate. From (1.3), (1.4) and (1.5) we observe that

$$\limsup_{h \to 0+} \frac{1}{h}\left[V(t+h, x+hf) - V(t, x) \right] \geqslant -\omega(t, x)$$

If $x(t)$ is any solutinn of (1.1), defining $m(t) = V(t, x(t))$, we get

$$\lim.\ \sup._{h \to 0+} \frac{1}{h}\left[m(t+h) - m(t) \right] \geqslant -\omega(t, m(t))$$

Now suppose that $b(t, \varepsilon)$ is a solution of

$$\rho' = -\omega(t, \rho) + \varepsilon,\ \varepsilon > 0, \text{ with } \rho(t_0) = \rho_0 > 0$$

Proceeding on the similar lines as in lemma I, we obtain the result of the lemma.

2.

Theorem : I

Let the assumptions of lemma I be satisfied and let $x(t)$ be continuous, satisfy $|x(t)| > 0$ and be a solution of (1.1) for $a \leqslant t \leqslant b$. Then for $a \leqslant t \leqslant b$, we have $|x(t)| \leqslant P^{-1}[P(|x(a)|) + (t-a)]$ (2.1)

$$\text{where } P(u) = \int_{u_0}^{u} [r(s)]^{-1}\, ds,\ u_0 \geqslant 0 \tag{2.2}$$

Proof : We have $\frac{dx}{dt} = f(t, x)$, so that

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds$$

so that

$$|x(t)| \leqslant |x(t_0)| + \int_{t_0}^{t} |f(s, x(s))|\, ds$$

$$\leqslant | x(t_0) | + \int_{t_0}^{t} V(s, x(s))\, ds$$

Set $y(t) = | x(t) |$, then we have

$$y(t) \leqslant y(t_0) + \int_{t_0}^{t} V(s, x(s))\, ds \tag{2.3}$$

Let

$$\psi(t) = y(t_0) + \int_{t_0}^{t} V(s, x(s))\, ds \tag{2.4}$$

then

$$\psi'(t) = V(t, x(t))$$

Then from (1.2) and (2.2) we have

$\frac{d}{dt} P(\psi(t)) \leqslant 1$, integration of which between (t_0, t) leads to

$P(\psi(t)) - P(\psi(t_0)) \leqslant t - t_0$

which gives

$\psi(t) \leqslant P^{-1}[P(\psi(t_0)) + (t - t_0)]$ where

$P^{-1}(u)$ is the inverse of $P(u)$. Since

$\psi(t_0) = y(t_0)$, we have

$\psi(t) \leqslant P^{-1}[P(y(t_0)) + (t - t_0)]$

and from (2.3) and (2.4), we get

$y(t) \leqslant P^{-1}[P(y(t_0)) + (t - t_0)]$.

Now changing t_0 to a we get (2.1).

Theorem 2:

Let the assumptions of lemma II be satisfied and let $x(t)$ be continuous, satisfy $|x(t)| > 0$ and be a solution of (1.1) for $a \leqslant t \leqslant b$. Then for $a \leqslant t \leqslant b$, we have

$$|x(t)| \leqslant Q^{-1}[Q(|x(a)|) + (t - t_0)] \tag{2.5}$$

where

$$Q(u) = -\int_{u_0}^{u} [\rho(s)]^{-1}\, ds \tag{2.6}$$

Proof:

We have

$$|x(t)| \geqslant |x(t_0)| - \int_{t_0}^{t} V(s, x(s))\, ds$$

Setting $y(t) = |x(t)|$, we get

$$y(t) \geqslant y(t_0) - \int_{t_0}^{t} V(s, x(s)) \tag{2.7}$$

Let

$$\phi(t) = y(t_0) - \int_{t_0}^{t} V(s, x(s)) \tag{2.8}$$

then $\phi'(t) = -V(t, x(t))$

From (1.6) and (2.6), we have

$\frac{d}{dt} Q(\phi(t)) \geqslant 1$, integration of which between (t_0, t) gives

$Q(\phi(t)) - Q(\phi(t_0)) \geqslant t - t_0$, from which we get

$$\phi(t) \geqslant Q^{-1}[Q(\phi(t_0)) + (t - t_0)]$$

Since $\phi(t_0) = y(t_0)$, we have

$$\phi(t) \geqslant Q^{-1}[Q(y(t_0)) + (t - t_0)]$$

From (2.7) and (2.8) we get

$$y(t) \geqslant Q^{-1}[Q(y(t_0)) + (t - t_0)].$$

Again changing t_0 to a we get (2.5).

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UNSTEADY FLOW OF A VISCOUS FLUID THROUGH A RECTANGULAR PIPE

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ABSTRACT

In the present note the technique of superposablity is adopted to solve the Navier-Stokes equation. Two solutions of the flow of a viscous liquid through a rectangular pipe are obatined. under different pressure gradients. The superposed solution is so adjusted that the velocity is zero for $t=o$. The particular case of the flow between two parallel planes is also discussed.

Recently Ram Ballabh has studied the unsteady flow of a viscous fluid through a circular pipe (1). He has used the method of superposability to obtain the solution which vanishes initially and also satifies the viscosity condition on the walls of the tube. In the present note we extend his method for a pipe of rectangular cross-section.

The axis of the cylinder is taken along the Z-axis and the velocity in that direction is denoted by U. From the equation of continuity it follows that u depends only on the x, y coordinates and the time t. We first obtain two particular solutions. These two solutions are superposed and the arbitrary constants are so adjusted that the velocity vanishes for $t=o$.

The boundary of any cross-section is taken as

$$(x^2 - a^2)(y^2 - b^2) = o \tag{0.1}$$

The velocity vanishes for $x = \mp a, y = \mp b$.

1. The equations of motion are

$$\nu\nabla^2 u - \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{1.1}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = o \tag{1.2}$$

From 1.2 we see that the fluid pressure is constant on any cross-section of the cylinder perpendicular to the axis. We consider two cases,

$$\text{(I)}\ \frac{\partial p}{\partial z} = o, \quad \text{(II)}\ \frac{\partial p}{\partial z} = a \text{ function of } t$$

Case (I) $\dfrac{\partial p}{\partial z} = o$

The equation 1.1 is solved assuming that the velocity contains an exponential decreasing time factor. We therefore take

$$u = e^{-\nu A^2 t} f(x,y) \tag{1.3}$$

The function $f(x,y)$ vanishes for $x = \mp a$ $y = \mp b$.
Substituting from 1.3 in 1.1 we have

$$(\nabla^2 + A^2) f(x,y) = 0 \tag{1.4}$$

with

$$f(\mp a, y) = 0, \; f(x, \mp b) = 0 \tag{1.5}$$

The regular solution of 1.4 satisfying 1.5 is

$$f(x,y) = C_{m,n} \cos \frac{(2m+1)\pi x}{2a} \cos \left(\frac{2n+1}{2b} \right) \pi y \tag{1.6}$$

and

$$A^2_{m,n} = \frac{(2m+1)^2\pi^2}{4a^2} + \frac{(2n+1)^2\pi^2}{4b^2}$$

The velocity is given by

$$u = C_{m,n}\, e^{-\nu A^2_{m,n} t} \cos \left(\frac{2m+1}{2a} \right) \pi x \cos \left(\frac{2n+1}{2b} \right) \pi y$$

The equation 1.1 being linear the solutions are superposable therefore u can be taken as

$$u = \Sigma \Sigma\, C_{m,n}\, e^{-\nu A^2_{m,n} t} \cos \left(\frac{2m+1}{2a} \right) \pi x \cos \left(\frac{2n+1}{2b} \right) \pi y \tag{1.7}$$

where the constants will have to be so chosen that the double series is uniformly convergent and can be differentiated term by term twice. The summation in 1.7 ranges over all positive integral values of m and n.

Case (II).

We take

$$\frac{\partial p}{\partial z} = B\rho\, e^{-\alpha t} \tag{1.8}$$

u now satisfies the equation

$$\nu \nabla^2 u - \frac{\partial u}{\partial t} = B\, e^{-\alpha t} \tag{1.9}$$

Let

$$u = e^{-\alpha t} F(x,y)$$

Substituting in 1.9 we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\alpha}{\nu} \right) F - \frac{B}{\nu} = 0 \qquad (1.10)$$

The function $F(x,y)$ satisfies the same boundary conditions as the function $f(x,y)$, given by the equation 1.5.

Expressed as a Fourier series valid over the range $|y| < b$ the constant in 1.10 is

$$B = \frac{4B}{\pi} \sum_{\pi=1}^{\infty} \frac{(-1)^n}{2n+1} \cos\left(\frac{2n+1}{2b} \right) \pi y \qquad (1.11)$$

We shall assume that the function $F(x, y)$ is of the form

$$F(x, y) = \frac{-16Bb^2}{\pi^3 \nu} \sum \frac{(-1)^n}{(2n+1)^3} \cos\left(\frac{2n+1}{2b} \right) \pi y$$

$$[1 + S(x)] \qquad (1.12)$$

where $S(x)$ satisfies the differential equation

$$S''(x) - \left(N^2 - \frac{\alpha}{\nu} \right) S(x) + \frac{\alpha}{\nu} = 0 \qquad (1.13)$$

with

$$4 b^2 N^2 = (2n+1)^2 \pi^2$$

$S(x)$ has to satisfy the boundary cinditions (in view of the boundary conditions satisfied by $F(x,y)$).

$$1 + S(a) = 0, \qquad 1 + S(-a) = 0 \qquad (1.14)$$

the solution of 1.13 is

$$S(x) = A_1 e^{\sqrt{(N^2 - \alpha/\nu)}\, x} + B_1 e^{-\sqrt{(N^2 - \alpha/\nu)}\, x} + \frac{\alpha}{\nu(N^2 - \alpha/\nu)} \qquad (1.15)$$

Determining the arbitrary constants A_1 and B_1 from 1.14 it is easy to obtain

$$S(x) = -\frac{N^2}{N^2 - \alpha/\nu} \frac{\cosh \sqrt{(N^2 - \alpha/\nu)}\, x}{\cosh \sqrt{(N^2 - \alpha/\nu)}\, a} + \frac{\alpha}{\nu N^2 - a} \qquad (1.16)$$

Thus the function $F(x,y)$ is given by

$$F(x,y) = \frac{-4B}{\pi\nu} \Sigma \frac{(-1)^n}{2(n+1)\ \beta^2} \cos\left(\frac{2n+1}{2b}\right)\pi y$$

$$\left(1 - \frac{\cos h\ \beta\ x}{\mathrm{c\ s}\, h\ \beta\ a}\right) \qquad (1.17)$$

where $\qquad \beta^2 = N^2 - \frac{\alpha}{\nu}$

The equation of motion being linear the two solutions are additive. The superposed solution is

$$u = \Sigma\Sigma\, C_{m,n}\, e^{-\nu A^2 mn\, t} \cos\left(\frac{2m+1}{2a}\right)\pi \times \cos\left(\frac{2n+1}{2b}\right)\pi y$$

$$-\frac{4B\, e^{-\alpha t}}{\pi\nu} \Sigma \frac{(-1)^n}{(2n+1)\beta^2} \cos\left(\frac{2n+1}{2b}\right)\pi y \left(1 - \frac{\cos h\ \beta x}{\cos\ h\ \beta a}\right) \quad (1.18)$$

2. *Evaluation of the Constants.*

We shall evoluate the coefficients $C_{m,n}$ by assuming the velocity to vanish at $t=0$. For $t=0$ we get from 1.18

$$u = \sum_{n=1}^{\infty} \cos\frac{(2n+1)\ \pi y}{2b}\left[\sum_{m=1}^{\infty} C_{m,n} \cos\left(\frac{2m+1}{2a}\right)\pi x\right.$$

$$\left. - \frac{4B\,(-1)^n}{\pi\nu\ (2n+1)\beta}\left(1 - \frac{\cos h\ \beta x}{\cos\ h\ \beta a}\right)\right] \qquad (2.1)$$

If this is to vanish we must have

$$\sum_{m=1}^{\infty} C_{m,n} \cos\left(\frac{2m+1}{2a}\right)\pi x = \frac{4B\,(-1)^n}{\pi\nu(2n+1)\beta}\left(1 - \frac{\cos h\ \beta x}{\cos\ h\ \beta a}\right) \quad (2.2)$$

The right-hand side vanishes for $x = \mp a$. The constants $C_{m,n}$ are given by

$$a\, C_{m,n} = \frac{8\ B\,(-1)^m}{\pi\nu\ (2n+1)\beta^2)} \int_0^a \left(1 - \frac{\cos h\ \beta x}{\cos\ h\ \beta a}\right) \cos\frac{(2m+1)\ \pi x dx}{2a}$$

Evaluating the integral we have

$$a\, C_{m,n} = \frac{8\, B\, (-1)^{m+n}}{\pi v\, (2n+1)\, \beta^2} \left(\frac{1}{M} - \frac{M}{M^2+\beta^2} \right) \tag{2.3}$$

where

$$2a\, M = (2m+1)\, \pi$$

The velocity is given by

$$u = \frac{8B}{\pi v} \Sigma\, \Sigma \frac{(-1)^{m+n}\, e^{-v\,(M^2+N^2)\,t}}{(2n+1) a\, \beta^2} \left(\frac{1}{M} - \frac{M}{M^2+\beta^2} \right)$$

$$\cos \left(\frac{2m+1}{2a} \right) \pi x \cos \left(\frac{2n+1}{2b} \right) \pi y$$

$$- \frac{4Be^{-\alpha t}}{\pi v} \Sigma \frac{(-1)^n}{(2n+1)\beta^2} \cos \left(\frac{2n+1}{2b} \right) \pi y \left(1 - \frac{\cos h\, \beta x}{\cos h\, \beta a} \right) \tag{2.4}$$

If $\alpha = 0$, we have

$$u = \frac{B}{\pi v} \Sigma\, \Sigma \frac{(-1)^{m+n}\, e^{-v(M^2+N^2)t}}{(2n+1)\, N^2 a} \left(\frac{1}{M} - \frac{M}{M^2+\beta^2} \right)$$

$$\cos \left(\frac{2m+1}{2a} \right) \pi x \cos \left(\frac{2n+1}{2b} \right) \pi y - \frac{16 B b^2}{\pi^3 v} \Sigma \frac{(-1)^n}{(2n+1)^3}$$

$$\cos \left(\frac{2n+1}{2b} \right) \pi y \left(1 - \frac{\cos h\, \beta x}{\cos h\, \beta a} \right)$$

As $t \to \infty$ this reduces to the well-known classical solution (2)

$$U = A \left[b^2 - y^2 + \frac{4}{b} \Sigma\, (-1)^{n+1}\, N^{-3}\ \text{Sech}\, Na \cos h\, Nx \cos Ny \right]$$

where A is a constant.

3. Flow Between Parallel Planes.

As a limiting case we consider the flow between two fixed parallel planes $y=b$, $y=-b$. Case (I) gives

$$v \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = 0 \tag{3.1}$$

u vanishes for $y=b, y=-b$. It is easy to verify that u is given by

$$u = \Sigma\, C_n\, e^{-\nu N^2 t} \cos\left(\frac{2n+1}{2b}\right)\pi y \tag{3.2}$$

Case (II), taking

$$u = e^{-\alpha t} F(y), \quad \frac{\partial p}{\partial z} = \rho B\, e^{-\alpha t}$$

the equation of motion gives

$$\frac{d^2F}{dy^2} + \frac{1}{\nu}(\alpha F - B) = 0 \tag{3.3}$$

which is satisfied if

$$F = \frac{B}{\alpha} + \frac{B_1}{\alpha} \cos \sqrt{\frac{\alpha}{\nu}}\, y$$

The boundary conditions give

$$O = B + B_1 \cos \sqrt{\frac{\alpha}{\nu}}\, b$$

so that

$$B = \frac{-B_1}{\cos \sqrt{\frac{\alpha}{\nu}}\, b}$$

Thus the velocity is given by

$$u = \frac{B}{\alpha}\, e^{-\alpha t}\left(1 - \frac{\cos \sqrt{\frac{\alpha}{\nu}}\, y}{\cos \sqrt{\frac{\alpha}{\nu}}\, b}\right) \tag{3.4}$$

The superposed solution is

$$u = \Sigma\, C_n\, e^{-\nu N^2 t} \cos\left(\frac{2n+1}{2b}\right)\pi y$$

$$+\frac{B}{\alpha}\, e^{-\alpha t}\left(1-\frac{\cos\sqrt{\frac{\alpha}{\nu}}\, y}{\cos\sqrt{\frac{\alpha}{\nu}}\, b}\right) \tag{3.5}$$

To evaluate the constants C_n we impose the condition that u is zero for $t=0$. For this we take

$$1-\frac{\cos\sqrt{\frac{\alpha}{\nu}}\, y}{\cos\sqrt{\frac{\alpha}{\nu}}\, b}=\Sigma\, A_n \cos\left(\frac{2n+1}{2b}\right)\pi y$$

where

$$b\, A_n = 2\,(-1)^n\left(\frac{1}{N}-\frac{N}{N^2-\frac{k^2}{b^2}}\right),\quad k=\sqrt{\frac{\alpha}{\nu}}\, b$$

At $t=0$ 3.5 gives

$$u=\Sigma\left[\, C_n+\frac{2B\,(-1)^n}{b\alpha}\left(\frac{1}{N}-\frac{N}{N^2-\frac{k^2}{b^2}}\right)\right]$$

If this is to vanish we choose

$$C_n=\frac{2B\,(-1)^n}{b\alpha}\left(\frac{N}{N^2-\frac{k^2}{b^2}}-\frac{1}{N}\right)$$

As $\alpha \to 0$ it is easy to show that

$$C_n=\frac{2B\,(-1)^n}{b\nu\, N^3} \tag{3.6}$$

For any other value of t the velocity is given by

$$u=\frac{2B}{b\alpha}\,\Sigma\,(-1)^n\left(\frac{N}{N^2-\frac{k^2}{b^2}}-\frac{1}{N}\right)e^{-\nu N^2 t}\cos\left(\frac{2n+1}{2b}\right)\pi y$$

$$+\frac{B}{\alpha}\left(1-\cos\sqrt{\frac{\alpha}{\nu}}\, y\ \mathrm{Sec}\sqrt{\frac{\alpha}{\nu}}\, b\right)e^{-\alpha t}$$

At $\alpha = 0$ this gives, from 3.6 and 3.3

$$U = \frac{2B}{\nu b} \sum \frac{(-1)^n \; e^{-\nu N^2 t}}{N^3} \cos\left(\frac{2n+1}{2b}\right) \pi y - \frac{B}{2\nu} (y^2 - b^2)$$

As $t \rightarrow \infty$ this gives the classical solution (3),

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A NOTE ON FUNCTIONALLY BOUNDEDNESS OF SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS

By

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ABSTRACT

In this note we shall derive some criteria, for the boundedness of solutions of $x = f(t,x)$ *as* $t \to \infty$ in a sufficiently general way so as to include the results of V. Lakshmikantham, B. Viswanatham and A. Wintner as special cases.

1. Let x and $f(x, t)$ be vectors with real components $(x_1(t), x_2(t), \ldots x_n(t))$ and $(f_1, f_2, \ldots f_n)$ respectively.

Consider the differential system

(1.1) $x' = f(x, t)$ $x(t_0) = 0$ $(' = d/dt)$ where $f(x, t)$ be continuous in the region $t_0 \leqslant t < \infty$, $|x| < \infty$

In this note, we shall derive some criteria for the boundedness of the solutions of (1.1) as $t \to \infty$, in a sufficiently general way so as to include the results of [3], [4], [5], as special cases.

We shall suppose that the function $\phi(x(t), t)$ and another function $\omega(r(t), t)$ respectively possess the following properties.

(A) $\phi(x, t)$ is a function defined for vector x and real t with nonnegative real values, which is continuous in (x, t) has one sided partial derivatives with respect to t and the components of x and $\phi(x, t) = 0$ implies $x = 0$. Let ϕ_t, $\phi_x > 0$ be partial derivatives of ϕ with respect to t and x respectively.

(B) $\omega(r, t)$ is continuous, non-negative function defined for $r \geqslant 0$, $t_0 \leqslant t < \infty$ and $\chi(t)$ is the maximal solution of the differetial epuation $r' = \omega(r, t)$ passing through the point $(t_0, 0)$.

The functional boundedness of solutions of differential equation (1.1) is defined as follows:

A solutions $x(t)$ of the differential equation (1.1) is said to be functionally bounded with respect to ϕ and L, if it satisfies $\displaystyle\lim_{t\to\infty} \frac{\phi(x, t)}{L(t)} < \infty$ where the function $L(t) > O$ be continous in the region $t_0 \leqslant t < \infty$.

We shall first prove the following:

Lemma—Assume that (A) and (B) hold and let the function $f(x, t)$ of the differential system (1.1) satisfy the condition (1.2) $\phi_t(x, t) + \phi_x(x, t). \mid f(x, t) \mid \leqslant \omega [\phi(x, t), t]$ then $\phi(x, t) \leqslant \chi(t)$

Proof:—Let $m(t) = \phi(x,t)$

Taking right hand derivatives with respect to t, we get

$$| m'(t) | \leqslant \phi_t(x, t) + \phi_x(x, t). \left| \frac{dx}{dt} \right|$$

$$| m'(t) | \leqslant \phi_t(x, t) + \phi_x(x, t) \, | f(x, t) |$$

By condition (1.2) $| m'(t) | \leqslant \omega \left[\phi(x, t), t \right]$

$$\leqslant \omega \left[m(t), t \right]$$

Suppose $\chi(t, \varepsilon)$ is the solution of $r' = \omega(r, t) + \varepsilon \quad r(t_o) = 0$ where ε is a positive quantity. The maximal solution of $r' = \omega(r, t) + \varepsilon$ is given by

$$\lim_{\rightarrow} \chi(t, \varepsilon) = \chi(t) \text{ [2]}$$

We shall first show that

$$m(t) \leqslant \chi(t, \varepsilon) \quad t_o \leqslant t < \infty$$

Suppose at a point $t > 0$, the inequality is not satisfied. Let $[t_o, t_1]$ be the interval where

$$m(t) \geqslant \chi(t, \varepsilon)$$

At t_o, we have $m(t_o) = \chi(t_o, \varepsilon)$

Hence by taking right hand derivatives at t_o, we obtain

$$m(t_o') \geqslant \chi'(t_o, \varepsilon)$$

From this obtain

$$\omega \, [t_o, m(t_o)] \geqslant \omega \, [t_o, \chi(t_o, \varepsilon)] + \varepsilon$$

This obviously a contradiction. Hence $m(t) \leqslant \chi(t, \varepsilon)$

Making $\varepsilon \to 0$, we get

$$m(t) \leqslant \chi(t)$$

Note: - If we put $\phi(x, t) = |x|$, we obtain the results of [3] and [4] which are the generalisations of Bellman's lemma c. f. [1].

Theorem 1:—Assume that (A) and (B) hold and let the function $f(x, t)$ of the differential system (1.1) satisfy the condition

(1.3) $\phi_t(|x|, t) + \phi_x(|x|, t). |f(x, t)| \leqslant \omega[\phi(x, t), t]$ and if $\chi(t) = O(L(t))$ *as* $t \to \infty$, then every solution of (1.1) is functionally bounded as $t \to \infty$. In particular if $\chi(t) = 0(L(t))$ as $t \to \infty$, then each component of every solution of (1.1) tend to zero as $t \to \infty$.

Proof:—Let $m(t) = \phi(|x|, t)$

Taking right hand derivatives with respect to t, we get

$$|m'(t)| \leqslant \phi_t(|x|, t) + \phi_x(|x|, t). |f(x, t)|$$

(*i.e.*) $$|m'(t)| \leqslant \omega[\phi(|x|, t), t]$$

$$\leqslant \omega[m(t), t]$$

By applying the lemma, we get

$$m(t) \leqslant \chi(t)$$

(*i e.*) $\phi(|x|, t) \leqslant \chi(t)$

As $t \to \infty$, and with the assumptions of the theorem follows the desired results.

Note:—The results of [3] are obtained by putting $\phi(x, t) = |x|$.

Theorem 2:—Assume that (A) and (B) hold and let $f(x, t)$ of the differential system (1.1) satisfy the condition

(1.4) $$\phi_t(|x_1 - x_2|, t) + \phi_x(x_1 - x_2|, t) . |f(x_1, t) - f(x_2, t)|$$
$$\leqslant \omega[\phi(|x_1 - x_2|, t), t]$$

where x_1 and x_2 are two solution of (1.1) suppose $\chi(t) = 0(L(t))$ as $t \to \infty$ then, if one solution of (1.1) is functionally bounded, then every solution vector is functionally bounded as $t \to$. In particular if $\chi(t) = 0(L(t))$ as $t \to \infty$, then every solution tends to the same functional limit as $t \to \infty$.

Proof:—Since x_1 *and* x_2 be the solutions of (1.1)

Writing $| v(t) | = | x_1 (t - x_2 (t) |$

$$| v' (t) | = | x_1' (t) - x_2' (t) | = | f(x_1 (t), t) - f(x_2 (t), t) |$$

Let $m (t) = \phi (| v | , t)$

Taking right hand derivatives with respect to t

$$| m' (t) | \leqslant \phi_t (| v | , t) + \phi_x (| v | , t). | f(x_1, t) - f(x_2, t) |$$

$$\leqslant \omega [\phi (| v | , t), t]$$

$$\leqslant \omega [m (t), t]$$

By applying lemma, we get

$$m (t) \leqslant \chi (t)$$

$$i.e. \quad \phi (| v | , t) \leqslant \chi (t)$$

As $t \to \infty$, and with the assumptions of the theorem yields the desired results.

If we put $\phi (x, t) = | x |$, we obtain the results of [3].

In a similar way, we can derive the theorems 3, 4, and 5 of [3] in sufficiently general way.

In the end, I wish to thank Dr. B. Viswanatham for his criticism during the preperation of this note.

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INTERFERENCE OF KETONES IN THE ESTIMATION OF PERFUMERY ALCOHOLS IN SYNTHETIC MIXTURES BY ACETYLATION. PART III.

By

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ABSTRACT

In the classical method of estimation of primary and secondary alcohols by acetylation with acetic anhydride and fused sodium acetate, the presence of acetophenone causes interference and gives higher results for esters of alcohol and lower results for acetophenone in the acetylated product. Previously this interference was reported to be due to the ketonic group of acetophenone with a methylene group of the perfumery alcohol. In continuation of previous work considering that perfumery alcohol and the enolic form of ketone undergo acetylation, formulae have been derived for the calculation of percentage of individual alcohols persent in a pentenary mixture consisting of two primary or secondary alcohols, an acid, one ester and one ketone.

Consider a mixture containing a, b, c, d and k percent of alcohols A, B, ester C, acid D and interfering ketone K respectively. It is assumed that their respective molecular weights are M_a, M_b, M_c, M_d, and M_k, the ester value of 100 per cent pure acetate of alcohol A and B and ester C being V_a, V_b and V_c respectively. Let V_d be the acid value of 100 percent acid D and v_1 and v_2 respectively represent the ester value of the mixture before and after acetylation and v_3 and v_4 be the acid value of the mixture before and after acetylation.

The mixture after acetylation with hot acetic anhydride and fused sodium acetate contains k' per cent of interfering ketone. On the basis of original mixture let k_1 and $(k-k_1)$ percent be unchanged ketone (ketonic form) and enolic form of the interfering ketone which gives ester having ester value V_k. Then, before acetylation $a+b+c+d+k=100$ ... ... (1)

$$c = \frac{100 v_1}{V_c} + \frac{v_1 M_c}{561 \cdot 04} \qquad \ldots \qquad \ldots \qquad (2)$$

$$d = \frac{100 v_3}{V_d} = \frac{v_3 M_d}{561 \cdot 04} \qquad \ldots \qquad \ldots \qquad (3)$$

$$k = \frac{100\, x_1}{x_k} \qquad \ldots \qquad \ldots \qquad (4)$$

Where x_1 and x_k are the respective oxime number of the mixture and pure ketono K. Then after acetylation with hot acetic anhydride and sodium acetate, the proportion of acetates of A and B, ester C, and acid D and acetate of enolic form of interfering ketone and ketone K in the ketone form would be as follows:

$$a\left(1+\frac{42 \cdot 01}{M_a}\right) : b\left(1+\frac{42 \cdot 01}{M_b}\right) : c : d : (k-k_1)\left(1+\frac{42 \cdot 01}{M_k}\right) : k_1$$

Then, after hot acetylation

$$V_2 = \frac{a\left(1 + \frac{42{\cdot}01}{M_a}\right) V_a + b\left(1 + \frac{42{\cdot}01}{M_b}\right) V_b + CV_c + (k - k_1)\left(1 + \frac{42{\cdot}01}{M_k}\right) V_k}{a\left(1 + \frac{42{\cdot}01}{M_a}\right) + b\left(1 + \frac{42{\cdot}01}{M_b}\right) + c + d + (k - k_1)\left(1 + \frac{42{\cdot}01}{M_k}\right) + k_1} \quad (5)$$

As $(M_a + 42{\cdot}01)\ V_a = (M_b + 42{\cdot}01)\ V_b = (M_k + 42{\cdot}01)\ V_k = 56104$
and $b = 100 - a - c - d - k$

$$V_2 = \frac{\frac{56104\,a}{M_a} + \frac{56104\,(100 - a - c - d - k)}{M_b} + \frac{56104\,(k - k_1)}{M_k} + 100v_1}{100 + \frac{42{\cdot}01\,a}{M_a} + \frac{42{\cdot}01\,(100 - a - c - d - k)}{M_b} + (k - k_1)\frac{42{\cdot}01}{M_k}} \quad (5_a)$$

$$100v_2 + \frac{42{\cdot}01av_2}{M_a} + \frac{42{\cdot}01\,(100 - a - c - d - k)\,v_2}{M_b} + (k + k_1)\frac{42{\cdot}01v_2}{M_k}$$

$$= \frac{56104a}{M_a} + \frac{56104}{M_b}(100 - a - c - d - k) + \frac{56104\,(k - k_1)}{M_k} + 100v_1$$

$$\text{or } \frac{56104a}{M_a} - \frac{56104a}{M_b} - \frac{42{\cdot}01av_2}{M_a} + \frac{42{\cdot}01av_2}{M_b}$$

$$= \frac{56104}{M_b}(100 - c - d - k) - \frac{56104\,(k - k_1)}{M_k} + 100v_2 + \frac{42{\cdot}01}{M_b}$$

$$(100 - c - d - k)\,v_2 - 100v_1 + (k - k_1)\frac{42{\cdot}01v_2}{M_k}$$

$$\text{or, } 56104a\left[\frac{1}{M_a} - \frac{1}{M_b}\right] - 42{\cdot}01av_2\left[\frac{1}{M_a} - \frac{1}{M_b}\right] = 100v_2 - 100v_1 -$$

$$\frac{(k - k_1)}{M_k}(56104 - 42{\cdot}01v_2) - \frac{(100 - c - d - k)}{M_b}(56104 - 42{\cdot}01v_2)$$

$$\text{or, } a\left[\frac{1}{M_a} - \frac{1}{M_b}\right](56104 - 42{\cdot}01v_2) = 100v_2 - 100v_1 - (56104 - 42{\cdot}01v_2)$$

$$\left[\frac{100 - c - d - k}{M_b} + \frac{(k - k_1)}{M_k}\right] \quad \dots \quad \dots \quad (6)$$

$$\text{or } a = \frac{M_a}{M_a - M_b}\left[(100 - c - d - k) + \frac{(k - k_1) M_b}{M_k} - \frac{(v_2 - v_1) M_b}{561{\cdot}04 - 0{\cdot}4201 v_2}\right] \quad \ldots \quad \ldots \quad (7)$$

Next, substituting $a = (100 - b - c - d - k)$ in equation (5)

$$v_2 = \frac{\dfrac{56104 (100 - b - c - d - k)}{M_a} + \dfrac{56104 b}{M_b} + CV_c + (k - k_1)\left(1 + \dfrac{42{\cdot}01}{M_k}\right) V_k}{100 + \dfrac{42{\cdot}01 (100 - b - c - d - k)}{M_a} + \dfrac{42{\cdot}01 b}{M_b} + \dfrac{42{\cdot}01 (k - k_1)}{M_k}}$$

$$\text{or } 100 v_2 + \frac{42{\cdot}01 (100 - b - c - d - k) v_2}{M_a} - \frac{42{\cdot}01 b v_2}{M_b} + \frac{42{\cdot}01 (k - k_1) v_2}{M_k}$$

$$= \frac{56104 (100 - b - c - d - k)}{M_a} + \frac{56104 b}{M_b} + 100 v_1 + \frac{56104 (k - k_1)}{M_k}$$

$$\text{or } \frac{(v_2 - v_1)}{561{\cdot}04 - 0{\cdot}4201 v_2} = \frac{(100 - b - c - d - k)}{M_a} + \frac{b}{M_b} + \frac{(k - k_1)}{M_k}.$$

$$\text{Hence, } b = \frac{M_b}{M_a - M_b}\left[(100 - c - d - k) + \frac{(k - k_1) M_a}{M_k} - \frac{(v_2 - v_1) M_a}{561{\cdot}04 - 0{\cdot}420 v_2}\right] \quad \ldots \quad \ldots \quad (8)$$

With the help of the equation (7) and (8) it is possible to determine the percentage of individual perfumery alcohol in the mixture provided the value of k_1 and $(k - k_1)$ which are respectively the percentage of total ketone and the enolic form of ketone in the original mixture. The percent of ketone, in the acetylated product

$$k' = \frac{100 k_1}{a\left(1 + \dfrac{42{\cdot}01}{M_a}\right) + b\left(1 + \dfrac{42{\cdot}01}{M_b}\right) + c + d + (k - k_1)\left(1 + \dfrac{42{\cdot}01}{M_k}\right) + k_1} \quad (9)$$

Since $a + b + c + d + k = 100$

$$k' = \frac{100 k_1}{100 + \dfrac{42{\cdot}01 a}{M_a} + \dfrac{42{\cdot}01 b}{M_b} + \dfrac{42{\cdot}01 (k - k_1)}{M_k}} \quad \ldots \quad \ldots \quad (10)$$

Hence,

$$100k' + \frac{42{\cdot}01ak'}{M_a} + \frac{42{\cdot}01bk'}{M_b} + \frac{42{\cdot}01kk'}{M_k} - \frac{42{\cdot}01k_1k'}{M_k} = 100k_1$$

$$\text{or } 100k_1 + \frac{42{\cdot}01k_1k'}{M_k} = 100k' + \frac{42{\cdot}01ak'}{M_a} + \frac{42{\cdot}01bk'}{M_b} + \frac{42{\cdot}01kk'}{M_k}$$

$$\text{or } 100k_1\left[1 + \frac{0{\cdot}4201k'}{M_k}\right] = 42{\cdot}01k'\left[\frac{a}{M_a} + \frac{b}{M_b} + \frac{k}{M_k}\right] + 100k'$$

$$= k'\left[42{\cdot}01\left(\frac{a}{M_a} + \frac{b}{M_b} + \frac{k}{M_k}\right) + 100\right]$$

$$\text{Hence } k_1 = \frac{k'\left[1 + 0{\cdot}4201\left\{\frac{a}{M_a} + \frac{(100 - a - c - d - k)}{M_b} + \frac{k}{M_k}\right\}\right]}{\left[1 + \frac{0{\cdot}4201k'}{M_k}\right]} \qquad (11)$$

The equation (11) enables one to calculate k_1 the percentage of ketone in the ketonic form in the original mixture provided a is eliminated from it.

The acid value of the acetylated mixture v_4

$$= \frac{dv_d}{a\left(1 + \frac{42{\cdot}01}{M_a}\right) + b\left(1 + \frac{42{\cdot}01}{M_b}\right) + c + d + (k - k_1)\left(1 + \frac{42{\cdot}01}{M_k}\right) + k_1}$$

$$= \frac{dv_d}{a + \frac{42{\cdot}01a}{M_a} + b + \frac{42{\cdot}01b}{M_b} + c + d + k + \frac{42{\cdot}01k}{M_k} - k_1 - \frac{42{\cdot}01k_1}{M_k} + k_1}$$

Since $a + b + c + d + k = 100$

$$v_4 = \frac{dv_d}{100 + 42{\cdot}01\left\{\frac{a}{M_a} + \frac{b}{M_b} + \frac{(k - k_1)}{M_k}\right\}}$$

$$= \frac{dv_d}{100 + 42{\cdot}01\left\{\frac{a}{M_a} + \frac{100 - a - c - d - k}{M_b} + \frac{(k - k_1)}{M_k}\right\}}$$

$$\text{or } dv_d = 100v_4 + \frac{42{\cdot}01av_4}{M_a} + \frac{42{\cdot}01av_4}{M_b} + \frac{42{\cdot}01\,(100 - c - d - k)v_4}{M_b} + \frac{42{\cdot}01\,(k - k_1)v_4}{M_k}$$

$$a\left\{\frac{1}{M_a} - \frac{1}{M_b}\right\} = \frac{dv_d}{42{\cdot}01v_4} - \frac{(100 - c - d - k)}{M_b} - \frac{(k - k_1)}{M_k} - \frac{100}{42{\cdot}01}$$

Now equation (10) can be rearranged as follows :

$$k_1 = \frac{k'\left[1 + 0{\cdot}4201\left\{\left(\frac{a}{M_a} - \frac{a}{M_b}\right) + \frac{(100 - c - d - k)}{M_b} + \frac{k}{M_k}\right\}\right]}{1 + \frac{0{\cdot}4201k'}{M_k}}$$

Substituting by the value of $\left(\frac{a}{M_a} - \frac{a}{M_b}\right)$ in the above

$$k_1 = \frac{k'\left[1 + 0{\cdot}4201\left\{\frac{dv_d}{42{\cdot}01v_4} - \frac{(100 - c - d - k)}{M_b} - \frac{(k - k_1)}{M_k} - \frac{100}{4201} + \frac{(100 - c - d - k)}{M_b} + \frac{k}{M_k}\right\}\right]}{1 + \frac{0{\cdot}421k'}{M_k}}$$

$$= \frac{k'\left[1 + 0{\cdot}4201\left\{\frac{dv_d - 100v_4}{42{\cdot}01v_4} + \frac{k_1}{M_k}\right\}\right]}{1 + \frac{0{\cdot}4201k'}{M_k}} \quad \ldots \quad \ldots \quad (12)$$

$$= \frac{k'\left[1 + 0{\cdot}4201\left\{\frac{dv_d - 100v_4}{42{\cdot}01v_4} + \frac{k_1}{M_k}\right\}\right]}{1 + \frac{0{\cdot}201k'}{M_k}} \quad \ldots \quad \ldots \quad (13)$$

The equation (13) would enable one to calculate k_1 the percentage of ketone in the ketonic form in the original mixture.

$$k_1 = \frac{k' \left[1 + \frac{0{\cdot}4201 dv_d}{42{\cdot}01 v_4} - 1 + \frac{0{\cdot}4201 k_1}{M_k} \right]}{1 + \frac{0{\cdot}4201 k'}{M_k}}$$

$$\text{or } k_1 = \frac{\left[\frac{k' dv_d}{100 v_4} + \frac{0{\cdot}4201 k_1 k'}{M_k} \right]}{1 + \frac{0{\cdot}4201 k'}{M_k}} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (14)$$

$$\text{or } k_1 = \frac{k' dv_4}{100 v_4} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15)$$

The equation (15) enables one to calculate k_1 from d the percentage of acid D having acid value V_a and k' percentage of ketone in the acetylated mixture having acid value v_4. Thus, knowing the percentage of enolic form of ketone in the mixture it is possible to calculate the percentage of the two individual alcohols A and B present on a penteznary system consisting of two alcohols, one acid, one ester and one interfering type of ketone.

Experimental work on the verification of these formulas is in progress.

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UPPER AND LOWER BOUNDS OF THE NORM OF SOLUTIONS OF NON-LINEAR VOLTERRA INTEGRAL EQUATIONS

By

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ABSTRACT

Consider the system of integral equations of the form

$$u(t) = \varphi(t) + \int_0^t A(t-s) f(s, u(s))\, ds$$

The existence theorems of such equations have been considered by J. A. Nohel [1962]. Assume that the solution of the above equation exists, we consider in this note the bounds of the solution in a sufficiently general way.

1. Consider the system of integral equations of the form

$$(1.1) \qquad u(t) = \varphi(t) + \int_0^t A(t-s) f(s, u(s))\, ds$$

under the assumptions

(i) φ and f are given vectors with n components each; A is a given $n \times n$ matrix defined on $0 \leqslant t < \infty$, $|u| < \infty$.

(ii) $\varphi(t)$, $A(t)$ are continuous functions in $0 \leqslant t < \infty$ and $f(t, u)$ is continuous in (t, u) for all u and for all t. Define $\| . \| = \Sigma \, |\cdot|$

The existence theorms of such equations have been considered in [5]. Assume that the solution of (1.1) exist, we consider in this note the bounds of the solution in a sufficiently general way.

We require the following lemma before we proceed further.

Lemma:—If

$$(1.2) \qquad z(t) \leqslant k(t) + \alpha \int_0^t \omega(s, z(s))\, ds$$

where $\alpha > 0$, some constant ; $\omega(z, t) \geqslant 0$ is continuous in both variables in a certain region R: $0 \leqslant t < \infty$, $z \geqslant 0$; and monotonic increasing in z for every fixed value of 't' ; $z(t)$ and $k(t)$ are continuous functions on the same range, then

$$z(t) \leqslant k(t) + M(t)$$

where $M(t)$ is the maximal solution of

(1.3) $z'(t) = \alpha\,\omega\,(t, z(t) + k(t))$

through the point (o. o)

Proof:—Put $r(t) = z(t) - k(t)$

then the ineqality (1.2) becomes

$$r(t) \leqslant \alpha \int_0^t \omega\,(s, r(s) + k(s))\,ds$$

Take $r(t)$ as the zero approximation to the solution of the differential equation (1.3) through (o, o) and set up the successive approximations recursively by

$$r_{k+1}(t) = \alpha \int_0^t \omega\,(s, r_k(s) + k(s))\,ds$$

we ean show by induction that these suecessive approximations form a monotonic increasing sequence, for suppose that $r_k \geqslant r_{k-1}$.

Then

$$r_{k+1}(t) - r_k(t) = \alpha \int_0^t \{\, \omega\,(s, r_k(s) + k(s)) - \omega\,(s, r_{k-1}(s) + k(s)) \,\}\,ds$$

$\geqslant$ o Since $\omega\,(t, z)$ is monotonic increasing in z

Therefore $r_{k+1}(t) \geqslant r_k(t)$

But the zero approximations $\leqslant$ first approximation. So, the successive approximations form a monotonic increasing function sequence, and since they are uniformly bounded and equi-continuous must converge uniformly to a solution $\psi(t)$. Further it is clear that $\psi(t)$ is a solution of the differential equation (1.3) through the point (o, o) and $r(t) \leqslant \psi(t)$.

Hence $r(t) \leqslant M(t)$ where $M(t)$ is the maximal solution of (1.3). This proves the lemma.

II. Now we are in a position to prove the following result:

Theorem 1:—Let

(i) the vector function f of the system (1.1) satisfy

$$\| f(t, u(t)) \| \leqslant \omega(t, \| u(t) \|)$$

where ω satisfies the conditions of the lemma

(ii) $\| A(t) \| < \alpha$ for all values of t and α is the same as before and let $u(t)$ satisfy $\| u \| > 0$ and be a solution of (1.1) in the interval $0 \leqslant t < \infty$. Then for all t in $0 \leqslant t < \infty$, we have

$$(2.1) \qquad \| u(t) \| \leqslant \| \varphi(t) \| + M(t)$$

and

$$(2.2) \qquad \| u(t) \| \geqslant \| \varphi(t) \| + m(t)$$

where $M(t)$ and $m(t)$ are the maximal and minimal solutions of

$$z'(t) = \pm \alpha \, \omega(t, z(t) + \varphi(t))$$

through the point (o, o), respectively.

Proof:—Let $u(t)$ be a solution of (1.1)

$$u(t) = \varphi(t) + \int_0^t A(t-s) f(s, u(s)) \, ds$$

$$\| u(t) \| \leqslant \| \varphi(t) \| + \int_0^t \| A(t-s) \| \, \| f(s, u(s)) \| \, ds$$

Thus is view of conditions (i) and (ii) of the theorem, we have

$$\| u(t) \| \leqslant \| \varphi(t) \| + \alpha \int_0^t \omega(s, \| u(s) \|) \, ds$$

By applying lemma, we obtain

$$\| u(t) \| \leqslant \| \varphi(t) \| + M(t)$$

To prove (2.2), we have to use essentially the same argument as in lemma, but now we have to consider the minimal solution of $z'(t) = -\alpha \, \omega(t, z(t) + \varphi(t))$ through (o, o) instead of maximal solution of $z'(t) = \alpha \, \omega(t, z(t) + \varphi(t))$ through (o, o). This completes the proof.

Remark:—The above theorem includes some of the results of [5]. Taking $\varphi(t) = k_1$, $A(t) = k_2$ for some constants $k_1, k_2 > o$ we obtain the results of V. Lakshmikanth [4] which includes the results of Bellman [1], Behari [2] and Langenhop [3] as special cases.

It should be noted that the results in this note require that the function $\omega(t, z)$ satisfy a monotone condition with respect to z. In the coresponding results in Ordinary Differential Equations as I have considered in [6], this requirment is not necessary. It appears unlikely that this condition can be dropped in the case of integral equations. Except for this restriction, our results contain those mentioned above for Ordinary Differential Equations as very special cases.

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SOME THEOREMS OF INTEGRAL TRANSFORMS

By

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ABSTRACT

In some of his previous papers the author has studied inter-relationships between various integral transforms. Continuing the study in this paper also, he has established the relationships of the Laplace Transform and its generalizations as given by Meijer and Varma, with the H- and the Y- Transforms. Four theorems in this connection have been obtained and they have been illustrated by suitable examples.

1. **Introduction** : The present paper is in continuation of some previous ones ((8) & (9)) by the author in which inter-relationships between various integral transforms have been studied.

We call (6)

$$g_1(p) = \int_0^\infty (px)^{\frac{1}{2}} \, \Upsilon_\nu \, (px) f(x) \, dx \tag{1·1}$$

the Υ—Transform of order ν of $f(x)$ and regard p as a positive real variable.

Its reciprocal is called the H – Transform (6) defined by

$$g_2(p) = \int_0^\infty (px)^{\frac{1}{2}} \, H_\nu \, (px) f(x) \, dx \tag{1·2}$$

We shall denote (1·1) and (1·2) symbolically as

$$g_1(p) \, \underset{\nu}{\overset{\Upsilon}{=\!=}} \, f(x)$$

$$g_2(p) \, \underset{\nu}{\overset{H}{=\!=}} \, f(x)$$

respectively.

The classical Laplace Transform *viz.*,

$$\phi(p) = p \int_0^\infty e^{-px} f(x) \, dx, \qquad R(p) > 0 \tag{1·3}$$

has been generalized by Varma (7) in the form

$$\phi_1(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}\, W_{k,m}(px) f(x)\, dx \quad R(p) > 0 \tag{1·4}$$

while its generalizations by Maijer ((3) & (4)) were given as

$$\text{(i)}\ \phi_2(p) = \sqrt{\frac{2}{\pi}}\cdot p\int_0^\infty (px)^{\frac{1}{2}}\, K_\nu(px) f(x)\, dx \tag{1·5}$$

$$\&\ \text{(ii)}\ \phi_3(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(px) f(x)\, dx, \quad R(p) > 0 \tag{1·6}$$

We shall represent (1·3) to (1·6) as

$$\phi(p) \doteq f(x)$$

$$\phi_1(p) \underset{k,m}{\overset{V}{=\!=}} f(x)$$

$$\phi_2(p) \underset{\nu}{\overset{K}{=\!=}} f(x)$$

$$\phi_3(p) \underset{k,m}{\overset{M}{=\!=}} f(x)$$

respectively.

It may be noted that (1·4), (1·5) and (1·6) reduce to (1·1) when (*i*) $k = -m+\frac{1}{2}$ (*ii*) $\nu = \pm\frac{1}{2}$ and (*iii*) $k = \pm m$ respectively.

We now establish the following theorems :

2. *Theorem 1.* If $x^{\mu - 3/2} f(x)$ and $\phi(x)$ belong to $L(0, \infty)$

and if
$$\psi(p) \doteq x^{\mu - 3/2} f(x)$$

$$\phi(p) \underset{\nu}{\overset{H}{=\!=}} f(x)$$

then

$$\psi(p) = -\frac{2\,\Gamma(\mu+\nu)}{\pi}\, p\int_0^\infty y^{\frac{1}{2}}(y^2+p^2)^{-\frac{1}{2}\mu}\, Q_{\mu-1}^{-\nu}\left[p(y^2+p^2)^{-\frac{1}{2}}\right]\phi(y)\, dy$$

where

$$R(p) > 0, \quad R(\mu) > |R(\nu)|, \quad -\tfrac{1}{2} < R(\nu) < \tfrac{1}{2} \tag{2·1}$$

Proof: Titchmarsh (1937) gave the inversion formula for (1·2) in the form

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}}\, \Upsilon_\nu\,(xy)\,\phi\,(y)\,dy,\; R\,(y) > 0,\; -\tfrac{1}{2} < R\,(\nu) < \tfrac{1}{2} \tag{2·2}$$

Substituting this value of $f(x)$ in

$$\psi\,(p) = p \int_0^\infty e^{-px} x^{\mu - 3/2} f(x)\,d\,x$$

and interchanging the order of integration which is permissible by Fubini's theorem under conditions stated, we obtain

$$\psi\,(p) = p \int_0^\infty \phi\,(y) \Big[\int_0^\infty e^{-px} x^{\mu - 3/2}\, (xy)^{\frac{1}{2}}\, \Upsilon_\nu\,(xy)\,dx \Big]\,dy \tag{2·3}$$

Evaluating the inner integral by (2), p. 105, we immediately get the theorem.

3. *Theorem 2*: If $f(\sqrt{x})$ and $\phi\,(x)$ belong to $L\,(0,\ \infty)$

and if

$$\psi\,(p) \;\underset{k,\,m}{\overset{M}{=\!=}}\; f(\sqrt{x})$$

$$\phi\,(p) \;\underset{\nu}{\overset{H}{=\!=}}\; f(x)$$

then

$$\psi\,(p) = (-1)^n\, 2^{\frac{1}{2}} \int_0^\infty G^{2,2}_{3,4}\left(\frac{y^2}{4\,p} \middle| \begin{matrix} k - m,\; k + m,\; -\frac{1}{4} - \frac{1}{2}\nu \\ \frac{1}{4} + \frac{1}{2}\nu,\; \frac{1}{4} - \frac{1}{2}\nu,\; 2k,\; -\frac{1}{4} - \frac{1}{2}\nu \end{matrix} \right) \phi\,(y)\,dy \tag{3.1}$$

$$R\,(5/2 - 2k \pm 2m \pm \nu) > 0,\; -1/2 < R\,(\nu) < 1/2,\quad R\,(p) > 0,\; R\,(y) > 0$$

Proof: Substituting the value of $f\,(\sqrt{x})$ from (2.2) in

$$\psi\,(p) = p \int_0^\infty e^{-\frac{1}{2}px}\, (px)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}\,(px)\, f\,(\sqrt{x})\,dx$$

and interchanging the order of integration and using [(2) p. 117]

$$\int_0^\infty x^{2\lambda}\, (xy)^{\frac{1}{2}}\, e^{-1/4\,x^2}\, W_{k,\mu}\,(\tfrac{1}{2}x^2)\, \Upsilon_\nu\,(xy)\,dx$$

$$= (-1)^m\, 2^\lambda\, G^{2,\,2}_{3,\,4}\left(\frac{y^2}{2} \middle| \begin{matrix} -\mu - \lambda,\; \mu - \lambda,\; l \\ h,\; k',\; k - \lambda - \frac{1}{2},\; l \end{matrix} \right)$$

$$h = \frac{1}{4} + \tfrac{1}{2}\nu,\ K' = \frac{1}{4} - \tfrac{1}{2}\nu,\ l = -\frac{1}{4} - \tfrac{1}{2}\nu$$

$$R(2\lambda \pm 2\mu \pm \nu) > -\frac{5}{2}, \quad R(p) > 0$$

$$R(y) > 0, \quad -\tfrac{1}{2} < R(\nu) < \tfrac{1}{2} \tag{3·2}$$

we obtain (3·1)

4. *Theorem 3* : If $x^{\lambda-1} f(x)$ and $\phi(x)$ belong to $L(0, \infty)$

and if $\psi(p) \underset{\mu}{\overset{K}{=\!=}} x^{\lambda-1} f(x)$

$$\phi(p) \underset{\nu}{\overset{\Upsilon}{=\!=}} f(x)$$

then

$$\psi(p) = \frac{2^{\lambda+\frac{1}{2}}}{\pi} \frac{\Gamma\left(1 + \frac{\nu+\lambda\pm\mu}{2}\right)}{\Gamma(\nu + {}^{3}/_{2})\, p^{\lambda+\nu+{}^{1}/_{2}}}$$

$$\times \int_0^\infty y^{\nu+{}^{3}/_{2}}\ {}_3F_2\left[\begin{matrix}1, 1 + \frac{\nu+\lambda\pm\mu}{2} \\ 3/2, \nu + 3/2\end{matrix} : -\frac{y^2}{p^2}\right] \phi(y)\, dy$$

$$R(\lambda + \nu) > |R(\mu)| - 2, \quad R(p) > 0, \quad R(y) > 0, \quad -\tfrac{1}{2} < R(\nu) < \tfrac{1}{2} \tag{4·1}$$

Proof : Inversion formula [(6)] for (1·1) gives

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} H_\nu(xy)\, \phi(y)\, dy \tag{4·2}$$

Substituting this in

$$\psi(p) = \sqrt{\frac{2}{\pi}}\, p \int_0^\infty (px)^{\frac{1}{2}} K_\mu(px)\, x^{\lambda-1} f(x)\, dx$$

and interchanging the order of integration as before, we obtain the result (4·1) on using [(2) p. 165)]

$$\int_0^\infty x^{\sigma-\frac{1}{2}} (xy)^{\frac{1}{2}} K_\mu(ax)\, H_\nu(xy)\, dx$$

$$= \frac{2^{\sigma}\ \pi^{-\frac{1}{2}}\ y^{\nu+3/2}}{a^{\nu+\sigma+2}} \cdot \frac{\Gamma\left(1+\frac{\nu+\sigma\pm\mu}{2}\right)}{\Gamma(\nu+3/2)}$$

$$\times\ {}_3F_2\left[\begin{matrix}1, 1+\frac{\nu+\sigma\pm\mu}{2} \\ 3/2,\quad \nu+3/2\end{matrix} : -\frac{y^2}{a^2}\right],$$

$$R(\sigma+\mu) > |R(\mu)| - 2. \qquad (4{\cdot}3)$$

Corollary : Putting $\mu = \pm\frac{1}{2}$, the theorem reduces to :

If $x^{\lambda-1}$ $f(x)$ and $\phi(x)$ belong to $L(0, \infty)$

and if $\psi(p) \doteq f(x)$

$$\phi(p) \underset{\nu}{\overset{\Upsilon}{-}} f(x)$$

then

$$\psi(x) = \frac{2^{\lambda+\frac{1}{2}}\ p^{-\lambda-\nu-\frac{1}{2}}}{\pi\ \Gamma(\nu+3/2)} \cdot \Gamma\left(1+\frac{\nu+\lambda\pm\frac{1}{2}}{2}\right)$$

$$\times \int_0^\infty y^{\nu+3/2}\ {}_3F_2\left[\begin{matrix}1, 1+\frac{\nu+\lambda\pm\frac{1}{2}}{2} \\ 3/2,\quad \nu+3/2\end{matrix} : -\frac{y^2}{p^2}\right] \phi(y)\,dy \qquad (4{\cdot}4)$$

$$R(\lambda+\nu) > -3/2,\quad R(p) > 0,\quad R(y) \geq 0,\quad -\tfrac{1}{2} < R(\nu) < \tfrac{1}{2}$$

5. *Theorem 4* : If $f(\sqrt{x})$ and $\phi(x)$ belong to $L(0, \infty)$

and if $\psi(p) \underset{k,\,m}{\overset{V}{=}} f(\sqrt{x})$

$$\phi(p) \underset{\nu}{\overset{\Upsilon}{=}} f(x)$$

then

$$\psi(p) = \frac{\pi^{-1/2}\ p^{-\nu/2-3/4}}{2^{2m+\nu+1}} \cdot \frac{\Gamma(7/4+\nu/2)\ \Gamma(7/4+\nu/2+2m)}{\Gamma(\nu+3/2)\ \Gamma(9/4+m-k-\nu/2)}$$

$$\times \int_0^\infty y^{\nu+3/2}\ {}_3F_3\left[\begin{matrix}1, \frac{7}{4}+\frac{\nu}{4}, \frac{7}{4}+\frac{\nu}{2}+2m \\ 3/2,\ \nu+3/2,\ \frac{9}{4}+m-k+\frac{\nu}{2}\end{matrix} : -\frac{y^2}{4p}\right] \phi(y)\,dy \qquad (5{\cdot}1)$$

$$R\left(\frac{7}{4}+\frac{\nu}{2}+2m\right)>0,\quad R(p)>0,\quad R(y)>0,\quad -\tfrac{1}{2}<R(\nu)<\tfrac{1}{2}.$$

Proof: Substituting for $f(\sqrt{x})$ from (4·2) in

$$\psi(p)=p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}\;W_{k,m}(px)\,f(\sqrt{x})\,dx$$

and inverting the order of integration, we get on using [(2), p. 171]

$$\int_0^\infty x^{2\lambda}(xy)^{1/2}\;e^{-1/4\,x^2}\,W_{k,\mu}\,(1/2\,x^2)\;H_\nu\,(xy)\,dx$$

$$=\frac{\pi^{-1/2}\quad 2^{1/4-\lambda-\nu/2}\quad\Gamma(7/4+\nu/2+\lambda\pm\mu)}{\Gamma(\nu+3/2)\;\Gamma(9/4+\lambda-k-\nu/2)}\;y^{\nu+3/2}$$

$$\times\;{}_3F_3\left[\begin{matrix}1,\ 7/4+\nu/2+\lambda\pm\mu\\ 3/2,\ \nu+3/2,\ 9/4+\lambda-k+\nu/2\end{matrix}\;:-\frac{y^2}{2}\right]$$

$$R(2\lambda+\nu)>2\,|\,R(\mu)\,|\,-\frac{7}{2}\tag{5·2}$$

the Theorem (5·1)

Corollary: Putting $K=-m+1/2$, the theorem reduces to :

If $f(\sqrt{x})$ and $\phi(x)$ belong to $L\,(0,\ \infty)$

and if $\psi(p)\doteq f(\sqrt{x})$

$$\phi(p)\underset{\nu}{\overset{Y}{=\!=}}\;f(x)$$

then

$$\psi(p)=\frac{\pi^{-1/2}\,p^{-\nu/2-3/4}}{2^{2m+\nu+1}}\cdot\frac{\Gamma(7/4+\nu/2)\;\Gamma(7/4+\nu/2+2m)}{\Gamma(\nu+3/2)\;\Gamma(7/4-\nu/2+2m)}$$

$$\times\int_0^\infty y^{\nu+3/2}\;{}_2F_2\left[\begin{matrix}1,\ 7/4+\nu/2\\ 3/2,\quad \nu+3/2\end{matrix}\;-:\frac{y^2}{4p}\right]\phi(y)\,dy\tag{5.3}$$

$$-1/2<R(\nu)<1/2,\quad R(7/4+\nu/2+2m)>0,\ R(p)>0,\ R(y)>0$$

6. The above theorems may be utilized to evaluate certain infinite integrals. By way of illustration we take a few examples.

Example 1: In theorem 1, taking

$$f(x)=x^{-1/2}\;k_{2\nu}\,(2\sqrt{ax})$$

and finding $\phi(p)$ [(2) p (168)] and $\psi(p)$ [(1) p. 199], it is easy to obtain

$$\int_0^\infty (y^2+p^2)^{-1/2\mu} \; Q_{\mu-1}^{-\nu}\left[p(y^2+p^2)^{-1/2}\right] S_{\nu-1,\nu}\left(\frac{a}{y}\right) dy$$

$$= -a \frac{-1/2 \;\; \pi^2 \; p^{-\mu+3/2}}{2^{\nu+2}} \frac{\Gamma(\mu-1\pm\nu)}{\Gamma(\nu+1)\,\Gamma(\mu+\nu)} \cdot W_{-\mu+3/2,\nu}\left(\frac{a}{p}\right)$$

$S_{\mu,\nu}(z)$ being the Lommel function, and

$$R(\mu-1\pm\nu) > 0, \quad -1/2 < R(\nu) < 1/2 \qquad (6\cdot1)$$

Example 2 : In theorem 3, if we take

$$f(x) = x^{-1/2} J_\nu\left(\frac{a}{x}\right)$$

and find its $\phi(p)$ [(2) p. 110] and $\psi(p)$ [(2) p. 142]

we derive

$$\int_0^\infty y^{\nu+1} \, {}_3F_2\left[\begin{matrix}1, 1+\dfrac{\lambda+\nu\pm\mu}{2} \\ 3/2, \;\; \nu+3/2\end{matrix} : -\frac{y^2}{p^2}\right]$$

$$\times\left[y_{2\nu}(\sqrt{2ay}) + \frac{2}{\pi} k_{2\nu}(\sqrt{2ay})\right] dy$$

$$= \frac{\pi^{1/2} \; a^{\lambda-1/2}}{2^{2\lambda+1}} \cdot \frac{\Gamma(\nu+3/2)}{\Gamma\left(1+\dfrac{\nu+\lambda\pm\mu}{2}\right)} \, p^{\lambda+\nu+3/2}$$

$$\times G_{0,4}^{3,0}\left(\frac{a^2 p^2}{4}\,\middle|\, \frac{1/2-\lambda+\mu}{2}, \frac{1/2-\lambda-\mu}{2}, \frac{1}{4}+\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}\right)$$

$$R(a) > 0, \quad R(\lambda) > |R(\mu)| - 3/2, \quad -1/2 < R(\nu) < 1/2 \qquad (6\cdot2)$$
$$R(\lambda+\nu\pm\mu) > -2$$

and finally,

Example 3 : In theorem 4, let us take

$$f(x) = x^{\nu+3/2} \, {}_2F_1\left[\begin{matrix}1 \;\; m-k+\dfrac{\nu}{2}+\dfrac{9}{4} \\ 3/2\end{matrix} : -x^2\right]$$

$\phi(p)$ is then given by (2), p. 118 and to find $\psi(p)$ we use Rathie's result (5), *viz.*;

$$\frac{\Gamma(\delta)\, p^{1-\gamma}}{\Gamma(\beta)\,\Gamma(\gamma+m-k+1/2)}\, E\left[\begin{matrix}\beta, \gamma, \gamma+2m \\ \delta\end{matrix} : p\right]$$

$$\frac{V}{\overline{k, m}}\; x^{\gamma-1}\; {}_2F_1\left[\begin{matrix}\beta, \gamma+m-k+1/2 \\ \delta\end{matrix} : -x\right]$$

$$R(\gamma) > 0, \quad R(\gamma+2m) > 0$$

so that we get on a slight simplification after putting

$$\frac{7}{4} + \frac{v}{2} = \alpha, \; \frac{7}{4} + \frac{v}{2} + 2m = \beta; \quad m - k - \frac{v}{2} + \frac{3}{4} = \gamma$$

$$\int_0^\infty y^{\gamma+4\alpha-5}\; {}_3F_3\left(\begin{matrix}1, \alpha, \beta \\ 2\alpha-2, \gamma+2\alpha-2, 3/2\end{matrix} : -\frac{y^2}{4p}\right] k_\gamma(y)\, dy$$

$$= \frac{\pi^{1/2}\;\Gamma(2\alpha-2)\,\Gamma(\gamma+3/2)}{\Gamma(\alpha)\,\Gamma(\beta)}\, 2^{\beta+3\alpha+\gamma-6}$$

$$\times E\left[\begin{matrix}1, \alpha, \beta \\ 3/2\end{matrix} : p\right]$$

$$R(\alpha) > 0, \quad R(\beta) > 0, \quad R(\gamma+\alpha) > 1. \tag{6·3}$$

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ON ANOTHER REPRESENTATION OF AN H-FUNCTION

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ABSTRACT

Recently Fox has defined an H-function by a Mellin Barne's type integral, involving product of Gamma-functions; and has proved it to be a symmetrical Fourier Kernel. He further points out that it is a most general function and needs further study. In 1958 the author gave the Mellin Barne's type integral representation of this H-function, and established its kernel property basing his proof on the L^2-theory of general transforms. Moreover the author gave for it another interesting representation in the form of a multiple integral involving product of Bassel's functions; thus connecting H-function to a well known function of mathematical physics. In a paper he denotes it by

$$\psi_{v_1,\dots,v_m,\ \lambda_1,\dots,\lambda_n}^{\xi_1,\dots,\xi_p}(x) \quad \text{where } m+n>p.$$

A comparision of the forms of H and ψ functions at once leads to the fact of their being identical. The ψ - function, mentioned above, is a subsequent developement of a kernel $\psi_{v_1,\dots,v_m,\lambda}(x)$ defined earlier. A number of interesting applications, which establish self-reciprocity of well known functions have been made by the author.

1. In a recent paper [1] Fox has defined a function

(1·1)

$$H(x)=1/2\pi i.\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\prod_{i=1}^{q}\Gamma(b_i+c_i s)\prod_{j=1}^{p}\Gamma(a_j-e_j s)\,x^{-s}}{\prod_{i=1}^{q}\Gamma(b_i+c_i-c_i s)\prod_{j=1}^{p}\Gamma(a_j-e_j+e_j s)}\,ds$$

where $c_i>0,\ i=1,\dots q\ ;\ e_j>0,\ j=1,\dots,p\ ;\ q\geqslant p+1$, and all the poles of the integrand in (1·1) are simple. He has proved in details that the function in (1·1) is a symmetrical Fourier Kernel. He further points out that it is a most general function which needs further study.

The object of this paper is to give for an H- function another representation in the form of a multiple-integral involving product of Bessel functions.

2. In a recent paper [2], I have defined a function

(2·1) $\psi_{v_1,\dots,v_m,\lambda}(x)$

$$=\frac{\sqrt{x}}{\mu}\int_0^\infty\!\!\dots\!\int_0^\infty y^{1/\mu-2}J_{v_1}(t_1)\dots J_{v_{m-1}}(t_{m-1})\,J_{v_m}\!\left(\frac{x}{t_1\dots t_{m-1}\,y}\right)J_\lambda\left(y^{1/\mu}\right)\frac{dy\ dt_1\dots dt_{m-1}}{t_1\dots t_{m-1}}$$

where

$\mu > 0$; $\nu_k + \frac{1}{2} > 0$, $k = 1\ 2, \ldots, m$; $\lambda + \frac{1}{2} > 0$.

I have established that the function defined in (2·1) is a symmetrical Fourier Kernel.

Later on [2], the function in (2·1) has been generalised and the generalised function has been defined in the form

$$(2{\cdot}2)\quad \psi^{\xi_1,\ldots,\xi_p}_{\nu_1\ldots,\nu_m,\lambda_1,\ldots,\lambda_n}(x) = \frac{\sqrt{x}}{R}\int_0^\infty \cdots \int_0^\infty \frac{\nu(t_1\ldots t_p/y_1\ldots y_n)}{z_1\ldots z_{m-1}} J_{\nu_m}\left(\frac{xt_1\ldots t_p}{y_1\ldots y_n\, z_1\ldots z_{m-1}}\right) \times$$

$$J_{\nu_1}(z_1)\cdots J_{\nu_{m-1}}(z_{m-1})\,(y_1)^{1/\mu_1-3/2} J_{\lambda_1}(y_1^{1/\mu_1})\cdots (y_n)^{1/\mu_n-3/2} J_{\lambda_n}(y_n^{1/\mu_n}) \times$$

$$(t_1)^{1/\mu'_1-\frac{1}{2}} J_{\xi_1}(t_1^{1/\mu'_1})\cdots(t_p)^{1/\mu'_p-\frac{1}{2}} J_{\xi_p}(t_p^{1/\mu'_p})\, dz_1\ldots dz_{m-1}\, dy_1\ldots dy_n\, dt_1\ldots dt_p$$

where $R = \mu_1\ldots\mu_n\,\mu'_1\ldots\mu'_p$ and

$\mu_k > 0$, $k = 1, 2 \ldots\ldots n$; $\mu'_r > 0$, $r = 1, 2, \ldots p$; $m \geqslant 1$, n and p are any positive integers or zero, provided that $m + n > p$. An empty product is taken as 1.

The ν's can be permuted amongst themselves but it is not permissible to permute either λ's or ξ's amongst themselves or amongst oneanother.

The Mellin Barne's type representation of this function has been obtained [2], in the form.

$$(2.3)\quad \psi^{\xi_1,\ldots,\xi_p}_{\nu_1,\ldots,\nu_m\,\lambda_1,\ldots,\lambda_n}(x)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{(s-\frac{1}{2})h} \prod_{r=1}^{m}\left\{\frac{\Gamma\left(\frac{\nu_r}{2}+\frac{s}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\nu_r}{2}-\frac{s}{2}+\frac{3}{4}\right)}\right\} \prod_{i=1}^{n}\left\{\frac{\Gamma\left(\frac{\lambda_i}{2}+\frac{\mu_i s}{2}-\frac{\mu_i}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{\lambda_i}{2}-\frac{\mu_i s}{2}+\frac{\mu_i}{4}+\frac{1}{2}\right)}\right\} \times$$

$$\times \prod_{j=1}^{p}\left\{\frac{\Gamma(\xi_j/2-\mu'_j s/2+\mu'_j/4+1/2)}{\Gamma(\xi_j/2+\mu'_j s/2-\mu'_j/4+1/2)}\right\} x^{-s}\, ds$$

where $h = m + \mu_1 + \ldots + \mu_n - \mu'_1 - \ldots - \mu'_p$ and

$m+n>p$. The formula in (2.3), unlike the formula in (2.2), is valid when $m \geqslant 0$. Also n and p may be any positive integers or zero. An empty product is taken as 1. Taking $m=0$, $n=q$ and replacing x by $x\,2^h$
(2.3) reduces to

$$(2.4)\quad 2^{h/2}\ \psi^{\xi_1,\ldots,\xi_p}_{\lambda_1,\ldots,\lambda_q}\ (x\,2^h)$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \prod_{i=1}^{q} \left\{ \frac{\Gamma\left(\frac{\lambda_i}{2}+\frac{\mu_i s}{2}-\frac{\mu_i}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{\lambda_i}{2}-\frac{\mu_i s}{2}+\frac{\mu_i}{4}+\frac{1}{2}\right)} \right\}$$

$$\prod_{j=1}^{p} \left\{ \frac{\Gamma\left(\frac{\xi_j}{2}-\frac{\mu'_j s}{2}+\frac{\mu'_j}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{\xi_j}{2}+\frac{\mu'_j s}{2}-\frac{\mu'_j}{4}+\frac{1}{2}\right)} \right\} x^{-s}\,ds$$

Thus with $b_i=\lambda_i/2-\mu_i/4+1/2$, $c_i=\mu_i/2$, $i=1,\ldots,q$ and $a_j=\xi_j/2+\mu'_j/4+1/2$, $e_j=\mu'_j/2, j=1,\ldots,p$; the function in (2·4) is identical to $H(x)$ of (1.1). I gave this formula as early as 1958 [2]. Likewise (2.3) can be deduced from (1.1) by a little adjustment of the values of the parameters. I have established independently [2], using the theory of convergence in mean square sense, that the function in (2.2) is a symmetrical Fourier Kernel.

Further let $m=1$, $\nu_1=\nu$, $n=q$ and replace x by $2^h x$, where 'h' has the value assigned above, then from (2.2) and (2.3) we have the equivalent forms.

$$(2\cdot5)\quad 2^{h/2}\ \psi^{\xi_1,\ldots,\xi_p,}_{\nu,\lambda_1,\ldots,\lambda_q}\ (2^h x)$$

$$=\frac{\sqrt{x}\ 2^h}{R}\int_0^\infty\!\!\cdots\!\int_0^\infty V(t_1\ldots t_p/y_1\ldots y_q)\ J_\nu\ (2^h x\ t_1\ldots t_p/y_1\ldots y_q)\ \times$$

$$\times\ (y_1)^{1/\mu_1-3/2}\ J_{\lambda_1}\ (y_1^{\ 1/\mu_1})\ldots(y_q)^{1/\mu_q-3/2}\ J_{\lambda_q}\ (y_q^{\ 1/\mu_q})\ \times$$

$$\times\,(t_1)^{1/\mu'_1-1/2}\ J_{\xi_1}(t_1^{\ 1/\mu'_1})\ldots(t_p)^{1/\mu'_p-1/2}\ J_{\xi_p}\ (t_p^{\ 1/\mu'_p})\ dy_1\ldots dy_q\,dt_1\ldots dt_p$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s-1/2}\ \frac{\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{\nu}{2}-\frac{s}{2}+\frac{3}{4}\right)} \prod_{i=1}^{q} \left\{ \frac{\Gamma\ (b_i+c_i s)}{\Gamma\ (b_i+c_i-c_i s)} \right\}$$

$$\prod_{j=1}^{p} \left\{ \frac{\Gamma\ (a_j-e_j s)}{\Gamma\ (a_j-e_j+e_j s)} \right\} x^{-s}\ ds$$

$$= H\ (2x)/\sqrt{2}$$

where q in (1.1) has been replaced by $(q+1)$ and we take $b_{q+1}=\nu/2+1/4$, $c_{q+1}=1/2$; and b_i, c_i

$i=1,\ldots q$; a_j, e_j, $j=1,\ldots, p$ have the values assigned to them in (2.4). Also h and R have values previously assigned to them.

We may also write (2.5) in a slightly different form by applying a result [3], 2.1, 17, p. 52), similar to Parseval theorem on Mellin transforms, in the form

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathrm{F}(s)\, \mathrm{G}(s)\, x^{-s}\, ds = \int_0^{\infty} g(u)\, f(x/u)\, du/u$$

where F (s) and G (s) are Mellin transforms of $f(x)$ and $g(x)$ respectively.

This gives another formula for the equivalent forms in (2.5) viz.

$$(2.6) \qquad 2^{h/2}\, \psi_{\nu,\ \lambda_1,\ldots,\lambda_q}^{\ \ \xi_1,\ldots,\xi_p} (2^h x) = \int_0^{\infty} H(x/y) \sqrt{y}\, J_\nu (y)\, dy/y,$$

where now H (x) has q factors in the first product as in (1.1) and not $q+1$ as in the defination of H (x) in (2.5). The values of b_i, c_i, $i=1,\ldots, q$ and a_j, $e_j, j=1,\ldots, p$ are the same as given in (2.4).

It is believed that this new representation of the H-function provides an extention, to its defination given by Fox, by representing it as a multiple integral involving the product of Bessel functions. Thus this representation connects H-function to a well known function of Mathematical Physics.

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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ABSTRACT

In this note we shall derive some criteria for the boundedness of solutions of partial differential equations of hyperbolic type of the form

$$z_{xy} = f(x, y, z, p, q)$$

We shall also consider when a solution of the differential equation is unbounded.

§ 1. In this note we shall derive some criteria for the boundedness of solutions of partial differential equations. We shall also consider when a solution of the differential equation is unbounded.

In § 2 we shall derive a lemma which is useful for subsequent discussion. In § 3 we shall find the estimate for the bound to the difference between any two solutions of the partial differential equation. In § 4 we consider a comparison theorem which follows naturally form the lemma proved in § 2.

Consider the initial value problem

$$\text{(A)} \qquad z_{xy} = f(x, y, z, p, q) \quad z(x, o) = \sigma(x),\ z(o, y) = \Gamma(y)$$

where $\sigma(o) = \Gamma(o) = z_0$ in the region R: $o \leqslant x < \infty,\ o \leqslant y < \infty$.

Suppose $f(x, y, z, p, q)$ is a real valued and continuous function defined on $o \leqslant x < \infty,\ o \leqslant y < \infty,\ -\infty < z, p, q < \infty$.

By a solution of (A), we mean a real valued continuous function $z(x, y)$ having partial derivatives $z_x(x, y)$, $z_y(x, y)$, $z_{xy}(x, y)$ in the region R. $z(x, y)$ satisfies the integral equation

$$\text{(1.1)} \qquad z(x, y) = \sigma(x) + \Gamma(y) - z_0 + \int_0^x \int_0^y f(s, t, z(s, t), z_x(s, t), z_y(s, t))\,ds\,dt$$

We shall say that the function $\varphi(x, y, z, p, q)$ possesses the property (*) if

$\varphi(x, y, z, p, q)$ is a continuous, non-negative and bounded function defined for $(x, y) \varepsilon R$ and non-negative z, p, q, non-decreasing in each of the variables z, p, q

Before proceeding further we shall derive a lemma

§ 2. *Lemma*: Let (i) $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$ be non-negative $\geq o$, measurable functions defined in R such that α is continuous, β is uniformely Lipschiz's continuous with respect to y and γ is uniformely Lipschiz's continuous with respect to x and satisfy

$$(2.1) \qquad \alpha(x,y) \leqslant \int_0^x \int_0^y \varphi(s, t, \alpha(s, t), \beta(s, t), \gamma(s, t))\, dsdt$$

$$(2.2) \qquad \beta(x, y) \leqslant \int_0^y \varphi(x, t, \alpha(x, t), \beta(x,t), \gamma(x, t))\, dt$$

$$(2.3) \qquad \gamma(x, y) \leqslant \int_0^x \varphi(s, y, \alpha(s, y), \beta(s, y), \gamma(s,y))\, ds$$

(ii) $\varphi(x, y, z, p, q)$ have the property (*) and $x(x,y)$ be the mazimal solution of

$$(B) \qquad \psi(x, y) = \int_0^x \int_0^y \varphi(s, t, \psi(s, t), \psi_x(s, t), \psi_y(s, t))\, dsdt$$

then $\qquad \alpha(x, y) \leqslant \chi(x, y)$

$\beta(x, y) \leqslant \chi_x(x, y)$

$\gamma(x, y) \leqslant \chi_y(x, y)$

Proof: Define the sequence of successive approzimations to (B) as follows: Let

$$(2.4) \qquad z^0(x, y) = \alpha(x,y),\ u^0(x, y) = \beta(x, y),\ v^0(x, y) = \gamma(x, y) \text{ and for } k \geqslant 1$$

$$(2.5) \qquad z^k(x, y) \leqslant \int_0^x \int_0^y \varphi(s, t, z^{k-1}(s, t), u^{k-1}(s, t), v^{k-1}(s, t))\, dsdt$$

$$(2.6) \qquad u^k(x,y) \leqslant \int_0^y \varphi(x, t, z^{k-1}(x, t), u^{k-1}(s, t), v^{k-1}(x, t))\, dt$$

$$(2.7) \qquad v^k(x,y) \leqslant \int_0^x \varphi(s, y, z^{k-1}(s, y), u^{k-1}(s, y), v^{k-1}(s, y))\, ds$$

The functions z^k, u^k, v^k are defined on R which can be taken independent of k.

The inequalities (2.1), (2.2), (2.3) gives the cases $k = o$ of

(2.8) $z^k \leqslant z^{k+1}, u^k \leqslant u^{k+1}, v^k \leqslant v^{k+1}$

The cases when $k > o$ follows by induction by virtue of monotony of φ. The boundedness of φ implies the uniform boundedness of the functions

z^k, u^k, v^k hence as $k \to \infty$

(2.9) $z = \lim z^k, u = \lim u^k, v = \lim v^k$ exists in R. It is obvious from (2.4), (2.8) and (2.9) that

(2.10) $$o < \alpha \leqslant z, o < \beta \leqslant u, o < \gamma \leqslant v$$

Lebesque's theorem on term by term integration under bounded convergence implies

(2.11) $$z(x, y) = \int_o^x \int_o^y \varphi(s, t, z(s, t), u(s, t), v(s, t))\, dsdt$$

(2.12) $$u(x, y) = \int_o^y \varphi(x, t, z(x, t), u(x, t), v(x, t))\, dt$$

(2.13) $$v(x, y) = \int_o^x \varphi(s, y, z(s, y), u(s, y), v(s, y))\, ds$$

It is clear that $z_x = u$, $z_y = v$ almost everywhere. Thus the lemma follows from (2.10) and with the assumption (ii) of the hypothesis.

§ 3. *Theorem* 1. Let (i) $\varphi(x, y, \psi, \psi_x, \psi_y)$ have the property (*).

(ii) the function f of the differential equation (A) satisfy the condition

$$| f(x, y, z^1, u^1, v^1) - f(x, y, z^2, u^2, v^2) | \leqslant \varphi(z, y, | z^1 - z^2 |, | u^1 - u^2 |, | v^1 - v^2 |)$$

where the functions z^1, z^2, u^1, u^2, v^1, and v^2 being arbitrary.

(iii) $\chi(x, y)$ be the maximal solution of (B), then the difference between the two solutions of (A) together with its partial derivatives w. r. t. x and y are $\leqslant \chi(x, y)$, $\chi_x(x, y)$, $\chi_y(x, y)$ respectively.

Proof: Let $z^1(z, y)$, $z^2(x, y)$ be two different solutions of (A).

Let $u^1(x,y)$, $v^1(x,y)$ and $u^2(x,y)$, $v^2(x,y)$ be the functions associated with z^1 and z^2. We have

$$|z^1(x,y) - z^2(x,y)| \leqslant \int_0^x\int_0^y |f(s,t,z^1,u^1,v^1) - f(s,t,z^2,u^2,v^2)|\,dsdt$$

$$\leqslant \int_0^x\int_0^y \varphi(s,t,|z^1-z^2|,|u^1-u^2|,|v^1-v^2|)\,dsdt$$

putting

$$\alpha = |z^1-z^2|$$

$$\beta = |u^1-u^2|, \qquad \gamma = |v^1-v^2|$$

we obtain

$$|z^1-z^2| \leqslant \int_0^x\int_0^y \varphi(s,t,\alpha,\beta,\gamma)\,dsdt$$

Similarly we get

$$|z_x^1-z_x^2| \leqslant \int_0^y \varphi(x,t,\alpha,\beta,\gamma)\,dt$$

$$|z_y^1-z_y^2| \leqslant \int_0^x \varphi(s,y,\gamma,\beta,\gamma)\,ds$$

By applying the lemma, we get

$$|z^1-z^2| \leqslant \chi(x,y)$$

$$|z_x^1-z_x^2| \leqslant \chi_x(x,y)$$

$$|z_y^1-z_y^2| \leqslant \chi_y(x,y)$$

This completes the proof of the theorem.

Corollary 1. If all the functions involved are defined throughout xy-plane and if $\chi(x,y)$, $\chi_x(x,y)$ $\chi_y(x,y)$ are bounded as $(x,y)\to\infty$ if any one solution of the differential equation (A) together with its partial derivatives w. r. t. x and y are known to be bounded, then every other solution together with its partial derivatives is also bounded.

Cor : 2. If in the above corollary we suppose that $f(x,y,o,o,o) = o$ then every solution together with its partial derivatives w. r. t. x and y is bounded.

Theorem 2: Let (1) $\varphi(x,y,\psi,\psi_x,\psi_y)$ have the property (*) and $\varphi(x,y,\psi,\psi_x,\psi_y) = 0$ if and only if $\psi = 0$, $\psi_x = 0$, $\psi_y = 0$.

(ii) the function f of the differential equation (A) satisfy the condition

$$| y (x, y, z^1, u^1, v^1) - f (x, y, z^2, u^2, v^2) | \geqslant \varphi (x, y, | z^1 - z^2 |, | u' - u^2 | \ | v^1 - v^2 |)$$

Then if any one solution of (B) together with its partial derivatives w. r. t. x and y is unbounded as $(x, y) \to \infty$, then at least one solution together with its partial derivatives w. r. t. x and y is unbounded.

Proof: Let z^1, z^2 be any solutions of (A). Let u^1, v^1 and u^2, v^2 be the functions associated with z^1 and z^2

$$z^1_{xy} - z^2_{xy} = f (x, y, z^1, u^1, v^1) - f (x, y, z^2, u^2, v^2),$$

putting

$$\alpha = z^1 - z^2$$

$$\beta = u^1 - u^2, \ \gamma = v^1 - v^2$$

Therefore

$$z^1_{xy} - z^2_{xy} = f (x, y, \alpha + z^2, \beta + u^2, \gamma + v^2) - f (x, y, z^2, u^2, v^2)$$

$$= F (x, y, \alpha, \beta, \gamma) \text{ (say)}$$

Hence by condition (ii)

$$| F (x, y, \alpha, \beta, \gamma) | \geqslant \varphi (x, y, | \alpha |, | \beta |, | \gamma |)$$

Since $F (x, y, \alpha, \beta, \gamma)$ is continuous and $\varphi (x, y, \alpha, \beta, \gamma) > 0$, the above inequality shows that α, β, γ are different from zero,

$F(x, y, \alpha, \beta, \gamma) \gtrless 0$ which implies that either

$$F (x, y, \alpha, \beta, \gamma) \geqslant \varphi (x, y, | \alpha |, | \beta |, | \gamma |)$$

$$\text{or } F (x, y, \alpha, \beta, \gamma) \leqslant - \varphi (x, y, | \alpha |, | \beta |, | \gamma |)$$

Let us consider the first case:

By similer argument of lemma in § 2 it can be easily verified that

$$\alpha (x, y) \geqslant k (x, y)$$

$$\beta (x, y) \geqslant k_x (x, y)$$

$$\gamma (x, y) \geqslant k_y (x, y)$$

where $k (x, y)$ is the given unbounded solution of (B). Hence the results follows immediately. Similarly we can prove the other case.

§ 4. In this section we shall compare the solutions of two differential equations.

Theorem 8. Suppose

$$z_{xy} = f (x, y, z, p, q); \ z (x, 0) = \sigma(x), \ z (0, y) = \Gamma (y)$$

$$\bar{z}_{xy} = g (x, y, \bar{z}, \bar{p}, \bar{q}); \ \bar{z} (x, 0) = \sigma(x), \ \bar{z} (0, y) = \Gamma (y)$$

where $\sigma(0) = \Gamma(0) = z_0$ are two differential equations in R, satisfying the follows conditions

(i) $|f(x, y, z, p, q) - g(x, y, \bar{z}, \bar{p}, \bar{q})|$

$$\leqslant \varphi(x, y, |z - \bar{z}|, |p - \bar{p}|, |q - \bar{q}|)$$

(ii) $\varphi(x, y, z, p, q)$ have the property (*) and $\chi(x, y)$ is the maximal solution of (B) and $\chi(x, y)$, $\chi_x(x, y)$, $\chi_y(x, y)$ are bounded as $(x, y) \to \infty$

(iii) Any solution of one of the equations together with its partial derivatives w. r. t. x and y is bounded then, any solution of the other equation together with its partial derivatives w. r. t. x and y is also bounded.

Proof : The solutions of the two equations are given by

$$z(x, y) = \sigma(x) + \Gamma(y) - z_0 + \int_0^x \int_0^y f(s, t, z, p, q)\, ds dt$$

$$\bar{z}(x, y) = \sigma(x) + \Gamma(y) - z_0 + \int_0^x \int_0^y g(s, t, \bar{z}, \bar{p}, \bar{q})\, ds dt$$

Therefore

$$|z - \bar{z}| \leqslant \int_0^x \int_0^y |f(s, t, z, p, q) - g(s, t, \bar{z}, \bar{p}, \bar{q})|\, ds dt$$

$$\leqslant \int_0^x \int_0^y \varphi(s, t, |z - \bar{z}|, |p - \bar{p}|, |q - \bar{q}|)\, ds dt$$

Putting $\alpha = |z - \bar{z}|$, $\beta = |p - \bar{p}|$, $\gamma = |q - \bar{q}|$

$$|z - \bar{z}| \leqslant \int_0^x \int_0^y \varphi(s, t, \alpha, \beta, \gamma)\, ds dt$$

Similarly $|p - \bar{p}| \leqslant \int_0^y \varphi(x, t, \alpha, \beta, \gamma)\, dt$

$$|q - \bar{q}| \leqslant \int_0^x \varphi(s, y, \alpha, \beta, \gamma)\, ds$$

By applying the lemma we get

$$|z - \bar{z}| \leqslant \chi(x, y)$$
$$|z_x - \bar{z}_x| \leqslant \chi_x(x, y)$$
$$|z_y - \bar{z}_y| \leqslant \chi_y(x, y)$$

and the desired result immediately follows from (ii) and (iii).

Corallary : If $g(x, y, o, o, o) = o$ and χ, χ_x, χ_y are bounded as $(x, y) \to \infty$ every solution of (A) with its partial derivatives w. r. t. x and y is bounded as $(x, y) \to \infty$.

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ON CHAIN OF INTEGRAL TRANSFORMS AND THE E-FUNCTION OF MACROBERT

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ABSTRACT

The author here establishes two theorems, in which various integral transform are connected through a chain of relations, after first proving two lemmas required in their development. Obtaining some interesting properties of the MacRobert E-Function, he utilizes them to evaluate certain infinite integrals involving that function.

1. Introduction : In this paper we deal with certain properties of MacRobert E-Eunction and utilize them to develop a chain of relations under various Integral Transforms. Some new integrals involving E-Function have also been evaluated.

Meijer (4) gave a generalization of the classical Laplace Transform

$$\phi(p) = p \int_0^\infty e^{-pt} f(t)\, dt, \quad R(p) > 0 \tag{1.1}$$

as

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(pt) f(t)\, dt, \quad R(p) > 0 \tag{1.2}$$

while Varma (8) generalize it in the form

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{m-\frac{1}{2}} W_{k,m}(pt) f(t)\, dt, \quad R(p) > 0 \tag{1.3}$$

The generalized Stieltjes transform of $f(t)$ of order λ is given by

$$\phi(p) = p \int_0^\infty (p+t)^{-\lambda} f(t)\, dt \quad |\arg p| < \pi \tag{1.4}$$

Throughout this note we shall represent (1.1), (1.2), (1.3) and (1.4) symbolically as

$$f(t) \doteqdot \phi(p)$$

$$f(t) \xrightarrow[k,m]{M} \phi(p)$$

$$f(t) \xrightarrow[k,m]{V} \phi(p)$$

and $f(t) \xrightarrow[\lambda]{p} \phi(p)$

respectively.

2. Before establishing the main theorems we shall prove the following lemmas :

Lemma 1. If $t^{-2} g(t)$ belongs to $L(0, \infty)$ and

$$\text{if } g(t) \xrightarrow[k,m]{M} f(p) \quad \ldots \quad \text{(i)}$$

$$f(t) \doteqdot \phi(p) \quad \ldots \quad \text{(ii)}$$

$$\text{and } \quad t^{\gamma-2} \phi(t) \doteqdot \lambda_1(p) \quad \ldots \quad \text{(iii)}$$

then

$$\lambda_1(p) = p^{1-\gamma} \int_0^\infty u^{-2} \, E\left[\begin{matrix} 2-k+m, 2-k-m, \gamma \\ 2-2k \end{matrix} : pu \right] g(u) \, du \tag{2.1}$$

provided $R(p) > 0$, $R(2-k \pm m) > 0$ and the integrals involved are convergent.

Proof : With conditions (i) and (ii) above Jaiswal (3) has shown that

$$\phi(p) = \frac{\Gamma(2-k \pm m)}{\Gamma(2-2k)} \, p \int_0^\infty u^{-2} \, {}_2F_1\left[\begin{matrix} 2-k+m, 2-k-m \\ 2-2k \end{matrix} : -\frac{p}{u} \right] g(u) \, du$$

provided $R(p) > 0$, $R(2-k \pm m) > 0$ and the integral on the right is convergent. Finding $t^{\gamma-2} \phi(t)$ and substituting its value in

$$\lambda_1(p) = p \int_0^\infty e^{-pt} \, t^{\gamma-2} \, \phi(t) \, dt$$

and interchanging the order of integration which is permissible under conditions stated (Fubini's theorem) we obtain

$$\lambda_1(p) = \frac{\Gamma(2-k \pm m)}{\Gamma(2-2k)} \, p \int_0^\infty u^{-2} \, g(u) \int_0^\infty e^{-pt} \, t^{\gamma-1} \, {}_2F_1\left[\begin{matrix} 2-k \pm m \\ 2-2k \end{matrix} : -\frac{t}{u} \right] dt \, du$$

Evaluating the latter integral by (2) p. 212, we obtain the result.

Lemma 2. : If $t^{-2} g(t)$ belongs to $L(0, \infty)$

$$\text{and if } \quad g(t) \xrightarrow[k,m]{M} f(p)$$

$$f(t) \doteqdot \phi(p)$$

$$t^{\gamma-2} \phi(t) \doteqdot \lambda_1(p)$$

$$\text{and } t^{\gamma_1} \lambda_1\left(\frac{1}{t}\right) \xrightarrow[k_1, m_1]{M} \lambda_2(p)$$

then

$$\lambda_2(p) = p^{1-\gamma-\gamma_1} \int_0^\infty u^{-2} \, E\left[\begin{matrix} 2-k \pm m, \gamma, \gamma+\gamma_1-k_1 \pm m_1 \\ 2-2k, \gamma+\gamma_1-2k_1 \end{matrix} : pu \right] g(u) \, du$$

$$R(\gamma) > 0,\ R(2-k\pm m) > 0,\quad R(\gamma+\gamma_1-k_1\pm m_1) > 0 \tag{2.2}$$

Proof: Using the result of Lemma 1, we substitute the value of $t^{\gamma_1}\lambda_1(1/t)$ in

$$\lambda_2(p) = p\int_0^\infty e^{-\frac{1}{2}pt}\,(pt)^{-k_1-\frac{1}{2}}\,W_{k_1+\frac{1}{2},m_1}(pt)\,t^{\gamma_1}\lambda_1(1/t)\,dt$$

and interchange the order of integration which is permissible as before, to obtain

$$\chi_2(p) = p\int_0^\infty u^{-2}g(u)\int_0^\infty t^{\gamma+\gamma_1-1}\,e^{-\frac{1}{2}pt}\,(pt)^{-k_1-\frac{1}{2}}\,W_{k_1+\frac{1}{2},m_1}(pt)$$

$$\times\ E\left[\begin{matrix}2-k\pm m,\gamma\\2-2k\end{matrix} : -u/t\right]dt.\,du.$$

We can evaluate the latter integral with the help of (5, p. 171)

$$\int_0^\infty e^{-\frac{1}{2}u}\,u^{m+\gamma-3/2}\,W_{k,m}(u)\,E\left[\begin{matrix}\alpha_1\ldots\ldots\alpha_r\\\beta_1\ldots\beta_s\end{matrix} : \frac{p}{u}\right]du$$

$$= E\left[\begin{matrix}\alpha_1\ldots\alpha_r,\ \gamma,\ \gamma+2m\\\beta_1\ldots\beta_s,\ \gamma+m-k+\frac{1}{2}\end{matrix} : p\right] \tag{A}$$

and obtain (2.2)

3. By repeated application of result (A) we are in a position to generalize the result obtained in (2.2) and to state it in the form given below.

Theorem 1: If $t^{-2}g(t)$ belongs to $L(0,\infty)$

and if $\quad g(t) \xrightarrow[k,m]{M} f(p)$

$$f(t) \doteqdot \phi(p)$$

$$t^{\gamma-2}\phi(t) \doteqdot \lambda_1(p)$$

$$t^{\gamma_1}\lambda_1\left(\frac{1}{t}\right) \xrightarrow[k_1 m_1]{M} \lambda(p)$$

$$t^{\gamma_2}\lambda_2\left(\frac{1}{t}\right) \xrightarrow[k_2,\,m_2]{M} \lambda_3(p)$$

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$$t^{\gamma_{n-2}}\lambda_{n-2}\left(\frac{1}{t}\right) \xrightarrow[k_{n-2},\,m_{n-2}]{M} \lambda_{n-1}(p)$$

$$\text{and } t^{\gamma_{n-1}} \lambda_{n-1}\left(\frac{1}{t}\right) \xrightarrow[k_{n-1},\ m_{n-1}]{M} \lambda_n(p)$$

then

$$\lambda_n(p) = p^{1-\lambda-\alpha_{n-1}}$$

$$\times \int_0^\infty u^{-2}\, E\left[\begin{matrix} 2-k\pm m,\ \gamma,\ \gamma+\alpha_r - k_r + m_r,\ \gamma+\alpha_r - k_r - m_r \\ 2-2k,\ \gamma+\alpha_r - 2k_r \end{matrix} : pu\right] g(u)\, du \tag{3.1}$$

where $\alpha_r = \sum_1^r \gamma_r$; $r = 1, 2, \ldots (n-1)$ and $R(\gamma) > o$,

$R(2-k\pm m) > o$, $R(\gamma+\alpha_r - k_r \pm m_r) > o$, $r = 1, 2, \ldots (n-1)$

Corollary : Taking $k = \pm m$, $k_r = \pm m_r$ the theorem reduces to :

$$\text{If } t^{-2} g(t) \varepsilon\ L(o, \infty)$$

$$\text{and } g(t) \doteqdot f(p)$$

$$f(t) \doteqdot \phi(p)$$

$$t^{\gamma-2} \phi(t) \doteqdot \lambda_1(p)$$

$$t^{\gamma_1} \lambda_1\left(\frac{1}{t}\right) \doteqdot \lambda_2(p)$$

$$\cdots\cdots\cdots\cdots$$

$$t^{\gamma_{n-1}} \lambda_{n-1}\left(\frac{1}{t}\right) \doteqdot \lambda_n(p)$$

then

$$\lambda_n(p) = p^{1-\gamma-\alpha_{n-1}} \int_0^\infty u^{-2}\, E\left[2, \gamma, \gamma+\alpha_r :: pu\right] g(u)\, du$$

provided

$$R(\gamma) > o,\ R(\gamma+\alpha_r) > o,\ \alpha_r = \sum_1^r \gamma_r\ ;\ r = 1, 2, \ldots (n-1) \tag{3.0}$$

Example : Taking $g(t) = t^{k+l+\delta-\frac{1}{2}}\ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} : 1 - \frac{1}{t}\right]$

its Meijer's transform is given by Verma (9, p. 172) as

$$f(p) = A\, p^{-l-k-\frac{1}{2}}\ G^{4,\,2}_{3,\,4}\left(p \,\middle|\, \begin{matrix} 1-\delta-a,\ 1-\delta-b,\ l-k+\frac{1}{2} \\ -\delta,\ c-a-b-\delta,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2} \end{matrix}\right)$$

$R(p) > o$, $R(c-a-b) > o$, and $A = \dfrac{\Gamma(c)}{\Gamma(a)\ \Gamma(c-a)\ \Gamma(c-b)\ \Gamma(b)}$

$R\,(l\pm m+a+\delta+\tfrac{1}{2})>0,\quad R\,(l\pm m+b+\delta+\tfrac{1}{2})>0$

The Laplace transform is then (2, p. 222)

$$\phi(p)=A\,p^{l+k+\frac{1}{2}}\cdot G^{4,\,3}_{4,\,4}\left(\frac{1}{p}\left|\begin{matrix}1-\delta-a,\ 1-\delta-b,\ l-k+\frac{1}{2},\ l+k+\frac{1}{2}\\ -\delta,\ c-a-b-\delta,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\end{matrix}\right.\right)$$

$R\,(l+k+\delta-\tfrac{1}{2})>0,\quad R\,(l+k+a+b\ \ \delta-c-\tfrac{1}{2})>0,\quad |\arg p|<5\pi/2$

$$=A\,p^{l+k+\frac{1}{2}}\ G^{3,\,4}_{4,\,4}\left(p\left|\begin{matrix}1+\delta,\ 1-c+a+b+\delta,\ \frac{1}{2}-l-m,\ \frac{1}{2}-l+m\\ \delta+a,\ \delta+b,\ \frac{1}{2}-l+k,\ \frac{1}{2}-l-k\end{matrix}\right.\right),$$

since (1, p. 209)

$$G^{m,\,n}_{p,\,q}\left(x^{-1}\left|\begin{matrix}\alpha_r\\ \beta_s\end{matrix}\right.\right)=G^{n,\,m}_{q,\,p}\left(x\left|\begin{matrix}1-\beta_s\\ 1-\alpha_r\end{matrix}\right.\right)\tag{B}$$

Finding L. T. of $t^{\gamma-2}\ \phi(t)$ we get

$$\lambda_1(p)=A\,p^{3/2-l-k-\gamma}$$

$$G^{3,\,5}_{5,\,4}\left(\frac{1}{p}\left|\begin{matrix}3/2-l-k-\gamma,\ 1+\delta,\ 1-c+a+b+\delta,\ \frac{1}{2}-l-m,\ \frac{1}{2}-l+m\\ \delta+a,\ \delta+b,\ \frac{1}{2}-l+k,\ \frac{1}{2}-l-k\end{matrix}\right.\right)$$

$R\,(\gamma+2k)<0,\quad R\,(\tfrac{1}{2}-l-k-\gamma-a-\delta)>0,\quad R\,(\tfrac{1}{2}-l-k-\gamma-b-\delta)>0$

Now the Meijer's transform of $t^{\gamma_1}\,\lambda_1\left(\frac{1}{t}\right)$ can be found by using Verma's result (9, p. 173), so that we have

$$\lambda_2(p)=A\,p^{3/2-l-k-\gamma-\gamma_1}$$

$$\times G^{3,\,7}_{7,\,5}\left(\frac{1}{p}\left|\begin{matrix}3/2-l-k-\gamma-\gamma_1+k_1\pm m_1,\ 3/2-l-k-\gamma,\ 1+\delta,\ 1-c+a+b+\delta,\ \frac{1}{2}-l\mp m\\ \delta+a,\ \delta+b,\ \frac{1}{2}-l+k,\ \frac{1}{2}-l-k,\ 3/2-l-k-\gamma-\gamma_1+2k_1\end{matrix}\right.\right)$$

$R\,(\gamma+\gamma_1+2k-k_1\pm m_1)>0,\quad R\,(\delta+a+l+k+\gamma+\gamma_1-k_1\pm m_1-\tfrac{1}{2})>0$

Proceeding similarly we get ultimately,

$$\lambda_n(p)=A\,p^{3/2\ \ l-k-\gamma-a_{n-1}}$$

$$G^{3,\,2n+3}_{2n+3,\,n+3}\left(\frac{1}{p}\left|\begin{matrix}3/2-l-k-\gamma-\alpha_r+k_r\pm m_r,\ 3/2-l-k-\gamma,\ 1+\delta,\\ 1-c+a+b+\delta,\ \frac{1}{2}-l-m,\ \frac{1}{2}-l+m\\ \delta+a,\ \delta+b,\ \frac{1}{2}-l\pm k,\ 3/2-l-k-\gamma-\alpha_r+2k_r\end{matrix}\right.\right)$$

Where $\alpha_r = \sum_{1}^{r} \gamma_r$, $r = 1, 2, 3 \ldots (n-1)$

$R(\delta + a + l - k + \gamma + \alpha_r - k_r \pm m_r - \frac{1}{2}) > 0$, $R(\gamma + 2k + \alpha_r - k_r \pm m_r) > 0$

$$r = 1, 2, \ldots (n-1)$$

Substituting the value of $\lambda_n(p)$ in theorem (3.1) we get after a little simplification and putting

$$\gamma + \alpha_r - k_r + m_r = a_r$$

$$\gamma + \alpha_r - k_r - m_r = b_r$$

$$\gamma + \alpha_r - 2k_r = c_r$$

$$\int_0^\infty t^{k+l+\delta-5/2} \, E\left[\begin{matrix} 2-k\pm m, \gamma, a_1, \ldots a_{n-1}, & b_1, \ldots b_{n-1} \\ 2-2k, c_1, \ldots\ldots c_{n-1} & \end{matrix} : pt\right]$$

$$\times \; {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} : 1 \frac{1}{t}\right] dt$$

$$= \frac{p^{\frac{1}{2}-l-k} \, \Gamma(c)}{\Gamma(a) \cdot \Gamma(c-a) \, \Gamma(c-b) \, \Gamma(b)} \times$$

$$G^{2,\, 2n+3}_{2n+3,\, n+3}\left(\frac{1}{p}\middle| \begin{matrix} 3/2-l-k-a_r, \; 3/2-l-k-b_r, \; 3/2-l-k-\gamma, 1+\delta, \\ 1-c+a+b+\delta, \frac{1}{2}-l\pm m \\ \delta+a, \delta+b, \frac{1}{2}-l\pm k, \; 3/2-l-k-c_r \end{matrix}\right) \tag{3.3}$$

$$r = 1, 2, 3 \ldots (n-1)$$

$R(p) > 0$, $R(c-a-b) > 0$, $R(l \pm m + a + \delta + \frac{1}{2}) > 0$, $R(l \pm m + b + \delta + \frac{1}{2}) > 0$

$R(\gamma + 2k) < 0$, $R(\delta + a + l + k + b - c - \frac{1}{2}) > 0$, $R(\frac{1}{2} - \gamma - l - k - \delta - a) > 0$

$R(\frac{1}{2} - l - k - \gamma - \delta - b) > 0$, $R(\delta + a + l + k + a_r - \frac{1}{2}) > 0$, $R(\delta + a + l + k + b_r - \frac{1}{2}) > 0$

$R(a_r + 2k) > 0$, $R(b_r + 2k) > 0$

4. *Theorem* 2 : If $t^{-\gamma} \phi(t)$ belongs to $L(0, \infty)$

$$\text{and if} \quad \phi(t) \xrightarrow[k,\, m]{V} p^{2-\gamma} f(p)$$

$$f(p) \xrightarrow[\lambda]{p} \psi(p)$$

$$t^{\gamma_1}\ \psi\left(\frac{1}{t}\right) \xrightarrow[k_1,\, m_1]{V} \lambda_1(p)$$

$$t^{\gamma_2}\lambda_1\left(\frac{1}{t}\right) \xrightarrow[k_2,\, m_2]{V} \lambda_2(p)$$

. . . .

. . . .

$$t^{\gamma_{n-1}}\ \lambda_{n-2}\left(\frac{1}{t}\right) \xrightarrow[k_{n-1},\, m_{n-1}]{V} \lambda_{n-1}(p)$$

and $$t^{\gamma_n}\ \lambda_{n-1}\left(\frac{1}{t}\right) \xrightarrow[k_n,\, m_n]{V} \lambda_n(p)$$

then

$$\lambda_n(p) = p^{1-\lambda-\alpha_n}\frac{1}{\Gamma(\lambda)}\int_0^\infty u^{-\gamma}\ E\left[\begin{matrix}\lambda, \gamma, \gamma+2m, & \lambda+\alpha_r, & \lambda+\alpha_r+2m_r \\ \gamma+m-k+\frac{1}{2}, & \lambda+\alpha_r+m_r-k_r+\frac{1}{2} & \end{matrix} : pu\right]\phi(u)\,d \tag{4.1}$$

where $\alpha_r = \sum_1^r \gamma_r$; $r=1, 2,\ldots n$. and $R(\lambda) > 0$, $R(\gamma) > 0$

$R(\gamma+2m) > 0$, $R(\lambda+\alpha_r) > 0$, $R(\lambda+\alpha_r+2m_r) > 0$,

Proof: With the first three conditions above Saxena (7, p. 349) has shown that

$$\psi(p) = \frac{p^{1-\lambda}}{\Gamma(\lambda)}\int_0^\infty u^{-\gamma}\ E\left[\begin{matrix}\lambda, \gamma, \gamma+2m \\ \gamma+m-k+\frac{1}{2}\end{matrix} : pu\right]\phi(u)\,du$$

$R(\lambda) > 0$, $R(\gamma) > 0$, $R(\gamma+2m) > 0$

Finding $t^{\gamma_1}\ \psi\left(\frac{1}{t}\right)$ and substituting its value in

$$\lambda_1(p) = p\int_0^\infty e^{-\frac{1}{2}pt}\ (pt)^{m_1-\frac{1}{2}}\ W_{k_1,\, m_1}(pt)\ t^{\gamma_1}\ \psi\left(\frac{1}{t}\right)dt$$

we get on changing the order of integration and using the result (A) above

$$\lambda_1(p) = \frac{p^{1-\gamma_1-\lambda}}{\Gamma(\lambda)}\int_0^\infty u^{-\gamma}\ E\left[\begin{matrix}\lambda, \gamma, \gamma+2m, \lambda+\gamma_1, \lambda+\gamma_1+2m_1 \\ \gamma+m-k+\frac{1}{2}, \lambda+\gamma_1+m_1-k_1+\frac{1}{2}\end{matrix} : pu\right]\phi(u)\,du$$

$R(\lambda) > 0$, $R(\gamma) > 0$, $R(\gamma+2m) > 0$, $R(\lambda+\gamma_1) > 0$ $R(\lambda+\gamma_1+2m_1) > 0$

Proceeding similarly we get finally,

$$\lambda_n(p) = \frac{p^{1-\lambda-\alpha_n}}{\Gamma(\lambda)} \int_0^\infty u^{-\gamma} \, E\left[\begin{matrix}\lambda, \gamma, \gamma+2m, \lambda+\alpha_r, \lambda+\alpha_r+2m_r \\ \gamma+m-k+\frac{1}{2}, \lambda+\alpha_r+m_r-k+\frac{1}{2}\end{matrix} : pu\right] \phi(u)\, du$$

where $\alpha_r = \sum_1^r \gamma_r$; $\gamma=1, 2\ldots n$. and $R(\lambda) > o$, $R(\gamma) > o$, $R(\gamma+2m) > o$
$R(\lambda+\alpha_r) > o$, $R(\lambda+\alpha_r+2m_r) > o$

Hence the theorem.

Corollary: Taking $k_r = -m_r + \frac{1}{2}$; $r=1, 2\ldots n$ the theorem reduces to;

If $t^{-\gamma} \phi(t)$ belongs to $L(o, \infty)$

and if $\phi(t) \xrightarrow[k,\, m]{V} p^{2-\gamma} f(p)$

$$f(t) \xrightarrow[\lambda]{p} \psi(p)$$

$$t^{\gamma_1} \psi\left(\frac{1}{t}\right) \doteqdot \lambda_1(p)$$

$$t^{\gamma_2} \lambda_1\left(\frac{1}{t}\right) \doteqdot \lambda_2(p)$$

$$t^{\gamma_n} \lambda_{n-1}\left(\frac{1}{t}\right) \doteqdot \lambda_n(p)$$

then

$$\lambda_n(p) = \frac{p^{1-\lambda-\alpha_n}}{\Gamma(\lambda)} \int_0^\infty u^{-\gamma} \, E\left[\begin{matrix}\lambda, \gamma, \gamma+2m, \lambda+\alpha_r \\ \gamma+m-k+\frac{1}{2}\end{matrix} : pu\right] \phi(u)\, du \tag{4.2}$$

where $\alpha_r = \sum_1^r \gamma_r$; $r=1, 2,\ldots n$; $R(\gamma+2m) > o$ and $R(\lambda+\alpha_r) > o$

$R(\lambda) > o$, $R(\gamma) > o$

Example: Take

$$\phi(t) = \frac{\Gamma(\frac{1}{2}+c-k-m)}{\Gamma(c)\,\Gamma(c-2m)} a^{\nu} t^{c-2m-1} \, {}_3F_2\left[\begin{matrix}\frac{1}{2}-\nu\pm\mu, \frac{1}{2}+c-k-m \\ c, c-2m\end{matrix} : -\frac{t}{a}\right]$$

so that (7, p. 349)

$$\psi(p) = \frac{p^{2m+\gamma-c-\nu-\lambda+1}}{\Gamma(\lambda)\,\Gamma(\frac{1}{2}-\nu\pm\mu)} G^{3,2}_{2,3}\left(ap \,\middle|\, \begin{matrix}1+c+\nu-2m-\gamma, 1+\nu \\ \lambda+c+\nu-\gamma-2m, \frac{1}{2}+\mu, \frac{1}{2}-\mu\end{matrix}\right)$$

$R(\lambda) > o$, $R(\gamma+2m-c) > o$, $R(\gamma+2m-c+\frac{1}{2}-\nu\pm\mu) > o$

Therefore

$$t^{\gamma_1}\ \psi\left(\frac{1}{t}\right)=\frac{t^{\lambda+\mu+c-2m-\gamma+\gamma_1-1}}{\Gamma(\lambda)\ \Gamma(\frac{1}{2}-\nu\pm\mu)}$$

$$\times\ G^{3,\,2}_{2,\,3}\left(\frac{a}{t}\left|\begin{matrix}1+c-2m+\nu-\gamma,\ 1+\nu\\ \lambda+\nu+c-2m-\gamma,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu\end{matrix}\right.\right)$$

$$=\frac{t^{\lambda+\nu+c-2m-\gamma+\gamma_1-1}}{\Gamma(\lambda)\ \Gamma(\frac{1}{2}-\nu\pm\mu)}$$

$$\times\ G^{2,\,3}_{3,\,2}\left(at\left|\begin{matrix}1-\lambda-\nu-c+2m+\gamma,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\mu\\ \gamma+2m-c-\nu,\ -\nu\end{matrix}\right.\right),$$

on using (B).

Finding its Varma-transform (6 p. 275), we get

$$\lambda_1(p)=\frac{p^{1-\lambda-\nu-c+2m+\gamma-\gamma_1}}{\Gamma(\lambda)\ \Gamma(\frac{1}{2}-\nu\pm\mu)}$$

$$\times\ G^{2,\,5}_{5,\,3}\left(\frac{a}{p}\left|\begin{matrix}1-\lambda-\nu-c+2m+\gamma-\gamma_1,\ 1-\lambda-\nu-c+2m+\gamma-\gamma_1-2m_1,\\ 1-\lambda-\nu-c+2m+\gamma,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\mu\\ \gamma+2m-c-\nu,\ -\nu,\ 1-\lambda-\nu-c+2m+\gamma-\gamma_1+k_1-m_1-\frac{1}{2}\end{matrix}\right.\right)$$

$|\arg p|<2\pi$, $R(\lambda+\gamma_1)>0$, $R(\lambda+\gamma_1+c-2m-\gamma+m_1\pm m_1)>0$

Proceeding similarly we obtain

$$\lambda_n(p)=\frac{p^{1-\lambda-\nu-c+2m+\gamma-\alpha_n}}{\Gamma(\lambda)\ \Gamma(\frac{1}{2}-\nu\pm\mu)}$$

$$\times\ G^{2,\,2n+3}_{2n+3,\,n+2}\left(\frac{a}{p}\left|\begin{matrix}1-\lambda+\gamma-c+2m-\nu-\alpha_r,\ 1-\lambda+\gamma-c+2m-\nu-\alpha_r-2m_r\\ 1-\lambda+\gamma-c+2m-\nu,\ \frac{1}{2}-\mu,\ \frac{1}{2}+\mu\\ \gamma+2m-c-\nu,\ -\nu,\ 1-\lambda+\gamma-c+2m-\nu-\alpha_r+k_r-m_r-\frac{1}{2}\end{matrix}\right.\right)$$

where, $\alpha_r=\sum_1^r\gamma$; $r=1,2,\ldots n$.

Substituting in the theorem we get after a little simplification and putting $\frac{1}{2}-\nu+\mu=a_1$, $\frac{1}{2}-\nu-\mu=a_2$, $\frac{1}{2}+c-k-m=a_3$

$c=b_1, c-2m=b_2, \lambda+\alpha_r = c_r; \lambda+\alpha_r + 2m_r = d_r$ and $\lambda+\alpha_r - m_r - k_r + \frac{1}{2} = e_r$

$$\int_0^\infty u^{b_2-\gamma-1}\, {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1\ b_2 \end{matrix} : -\frac{u}{a}\right] E\left[\begin{matrix} \lambda, \gamma, b_1-b_2, c_r, d_r \\ \gamma+a_3-b_2, e_r \end{matrix} : pu\right] du$$

$$= \frac{\Gamma(b_1)\ \Gamma(b_2)\ a^{\frac{1}{2}(a_1+a_2-1)}}{\Gamma(a_1)\ \Gamma(a_2)\ \Gamma(a_3)}\ p^{\frac{1}{2}(a_1+a_2-1)+\gamma-b_2}$$

$$\times G^{2,\,2n+3}_{2n+3,\,n+2}\left(\frac{a}{p}\left|\begin{matrix} \frac{1}{2}(a_1+a_2+1)+\gamma-b_2-c_r, \frac{1}{2}(a_1+a_2+1)+\gamma-b_2-d_r, \\ \frac{1}{2}(a_1+a_2+1)+\gamma-b_2, \frac{1}{2}(1-a_1+a_2), \frac{1}{2}(1+a_1-a_2) \\ \frac{1}{2}(a_1+a_2-1)+\gamma-b_2, \frac{1}{2}(a_1+a_2+1), \frac{1}{2}(a_1+a_2+1) \\ +\gamma-b_2-e_r \end{matrix}\right.\right) \tag{4.3}$$

where $\quad a_r = \sum_1^r \gamma_r; \quad r=1, 2, 3 \ldots n.$

$R(\lambda) > o, R(b_1-b_2) > o, R(\gamma) > o, R(\gamma-b_2) > o$

$R(a_1-b_2+\gamma) > o, R(a_2-b_2+\gamma) > o, R(c_r) > o, R(d_r) > o$

$R(b_2+c_r-\gamma) > o, R(b_2+d_r-\gamma) > o$ and $|\arg p| < 2\pi$

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